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A TREATISE

ON THE

DYNAMICS OF A PARTICLE.



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DYNAMICS OF A PARTICLE

WITH NUMEROUS EXAMPLES.

BY

PETER GUTHRIE TAIT, M.A.,

LATE FELLOW OF ST PETER'S COLLEGE,
PROFESSOR OF NATURAL PHILOSOPHY IN THE UNIVERSITY OF EDINBURGH

AND THE LATE

WILLIAM JOHN STEELE, B.A.,

FELLOW OF ST PETER'S COLLEGE.

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PREFACE.

To the first edition of this work, published in 1856, the following was prefixed:—

"In the present Treatise will be found all the ordinary propositions, connected with the Dynamics of particles, which can be conveniently deduced without the use of D'Alembert's Principle.

"Its publication has been delayed by many unforeseen occurrences; more especially by the early and lamented death of Mr STEELE, whose portion of the work was left uncompleted, and whose assistance in its final arrangement and revision would have been invaluable. The principal portions due to him are the greater part of Chapters III., V. and VIII. together with a few pages of Chapter I.

"Considerable use has been made of Pratt's *Mechanical Philosophy:* indeed a large portion of Chapter XI. is reprinted verbatim from that work.

"Throughout the book will be found a number of illustrative examples introduced in the text, and for the most part completely worked out; others with occasional solutions or hints to assist the student are appended to each Chapter. For by far the greater portion of these the Cambridge Senate-House and College Examination Papers have been applied to."

I am glad of the opportunity, presented by the call for a second edition, to make reparation for many of the faults of the first. Numerous trivial errors, and a few of a more serious character, have now been corrected; many sections and several new examples have been added; and the whole of the second Chapter has been rewritten, upon the basis of the corresponding portion of Thomson and Tait's *Natural Philosophy* which, though as yet unpublished, was printed off nearly two years ago.

When I wrote that Chapter, in 1855, I had not read Newton's admirable introduction to the Principia; and I endeavoured to make the best of the information I had then acquired from English and French treatises on Mechanics. These five pages, faulty and even erroneous as I have since seen them to be, cost me almost as much labour and thought as the utterly disproportionate remainder of my contributions to the volume. And I cannot but ascribe this result, in part at least, to the vicious system of the present day, which ignores Newton's Third Law of Motion, though constantly assuming it (tacitly) as an axiom; and erects Statics upon a separate basis from Kinetics, thereby necessitating several additional Physical Axioms, the splitting of Newton's Second Law into two, and the introduction of a so-called *Statical* measure of Force.

To be enabled to preserve the title of the work, I have added (apropos of the Second Law of Motion) a few hints about Statics of a particle.

The Examples are, for the most part, reprinted *verbatim* from the papers in which they were set; in a few the language has been altered, or the theorem involved has been generalized; several, however, have defied all attempts at improvement, and now stand in their unintelligibility as a

warning, to the Candidate for Mathematical Honors, of the ordeal he may have to pass through.

To several important theorems more than one demonstration has been appended: with the object of exhibiting the use of the various processes by applying them to the deduction of results of real value, instead of to the solution of "Problems" of unquestionable absurdity.

Various friends to whom I have applied for suggestions as to any important changes which they might think desirable in this second edition, and especially I. TODHUNTER, Esq. of St John's, have replied that they had none to offer, as they liked the book well enough in its original form. This has prevented me from attempting a thorough alteration of style which I had contemplated, viz. to cease breaking up the subject into detached propositions—specially fitted for "writing out." I retain my own opinion, however, that this is *not* the form in which such a treatise ought to be written; although there can be no doubt that it offers certain advantages to the student whose sole object in reading is to pass an examination.

The treatise is intended to be merely an analytical one: for the full discussion and experimental demonstration of the elementary principles on which the analysis is founded, the reader must be referred to works on Natural Philosophy; of which, so far as mere Abstract Dynamics is concerned, we have a most admirable example in the Principia. For the general application of modern theories to the whole range of physical phenomena, the reader is referred to the forthcoming work on Natural Philosophy by Professor W. THOMSON and myself, in which the subject will be developed from the grand basis of Conservation of Energy.

I have been dissuaded from introducing into this work the

Newtonian notation for Fluxions. It is true that in Kinetics of a *particle* it is not very greatly superior to the ordinary notation of differential coefficients: though, when the general equations of motion of a system have to be treated, in the beautiful manner invented by Lagrange, a partial use of it is absolutely necessary. Newton's idea of Fluxions was purely Kinematical; and, in fact, the fundamental ideas of the Differential Calculus are essentially involved in the most elementary considerations regarding velocity. It is also to be observed, that, whenever we write $f'(x)$ for the differential coefficient of $f(x)$, we are really employing the principal feature of Newton's notation, though in a form somewhat more expressive than his.

It is possible that in this edition a few of the objectionable terms or methods, which the first edition contained, may have remained undetected—but I hope that in every essential respect the volume will be found to be an improvement on its predecessor.

I am encouraged in this hope by the fact that the sheets in passing through the press have been read by J. STIRLING, Esq. of Trinity, to whose care and knowledge I am indebted for many valuable suggestions.

P. GUTHRIE TAIT.

COLLEGE, EDINBURGH,
March, 1865.

CONTENTS.

The 13 pages are lacking
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spectively. But the actual velocity of the point is not greater than v_1, and not less than v_2, therefore as regards the actual space described,

$$\delta s \text{ is not greater than } v_1\delta t, \text{ and not less than } v_2\delta t,$$

$$\text{or } \frac{\delta s}{\delta t} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad v_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad v_2,$$

however small δt may be. But, as δt continually diminishes, v_1 and v_2 tend continually to, and ultimately become each equal to, v. Therefore, proceeding to the limit,

$$\frac{ds}{dt} = v.$$

If v be negative in this expression, it indicates that s diminishes as t increases; the positive case, which we have taken as the standard one, referring to that in which s and t increase together. It follows that, if a velocity in one direction be considered positive, in the opposite direction it must be considered negative; and consequently the sign of the velocity indicates the direction of motion.

8. It will be easily seen that the idea of velocity explained above is equally applicable whether the point be considered as moving in a straight, or in a curved, line. In the latter case, however, the direction of motion continually changes; and it will be necessary to know at every instant the direction, as well as the magnitude, of the point's velocity. This is done by considering the velocities of the point parallel to the three co-ordinate axes respectively. For, if the co-ordinates of the point be represented by x, y, z, the rates of increase of these, or the velocities parallel to the corresponding axes, will by reasoning analogous to the above be

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \frac{dz}{dt}.$$

Denoting by v the whole velocity of the point, we have

$$v = \frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}};$$

and, if α, β, γ be the angles which the direction of motion makes with the axes,

$$\cos\alpha = \frac{dx}{ds} = \frac{\frac{dx}{dt}}{\frac{ds}{dt}};$$

$$\text{or } \frac{dx}{dt} = v\cos\alpha = v_x, \text{ suppose.}$$

Similarly, $$\frac{dy}{dt} = v\cos\beta = v_y,$$

$$\frac{dz}{dt} = v\cos\gamma = v_z.$$

Hence, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ are the resolved parts of v parallel to the axes, and are therefore called the *Component Velocities* of the point: and, with reference to them, v is called the *Resultant Velocity*.

9. It follows from the above, that, if a point be moving in any direction, we may suppose its velocity to be the resultant of three coexistent velocities in any three directions at right angles to each other; or, more generally, in any three directions not coplanar. But the rectangular resolution is the simplest and best except in some very special questions.

Let v_x, v_y, v_z be the rectangular components of the velocity v of a moving point, then the resolved part of v along a line inclined at angles λ, μ, ν to the axes will be

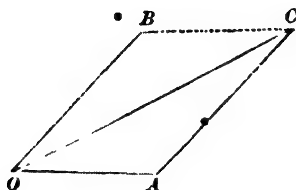
$$v_x\cos\lambda + v_y\cos\mu + v_z\cos\nu.$$

For, let α, β, γ be the angles which the direction of the point's motion makes with the axes, θ the angle between this direction and the given line. Then the resolved part of v along that line is

$$v\cos\theta = v\{\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu\}$$
$$= v_x\cos\lambda + v_y\cos\mu + v_z\cos\nu.$$

10. These propositions are virtually equivalent to the following obvious geometrical construction:—

To compound any two velocities as OA, OB in the figure; where OA, for instance, represents in magnitude and direction the space which would be described in one second by a point moving with the first of the given velocities—and



similarly OB for the second; from A draw AC parallel and equal to OB. Join OC:—then OC is the resultant velocity in magnitude and direction. For the motions parallel to OA and OB are independent.

OC is evidently the diagonal of the parallelogram two of whose sides are OA, OB.

Hence the resultant of any two velocities as OA, AC, in the figure is a velocity represented by the third side, OC, of the triangle OAC.

Hence if a point have, simultaneously, velocities represented by OA, AC, and CO, the sides of a triangle *taken in the same order*, it is at rest.

Hence the resultant of velocities represented by the sides of any closed polygon whatever, whether in one plane or not, taken all in the same order, is zero.

Hence also the resultant of velocities represented by all the sides of a polygon but one, taken in order, is represented by that one taken in the opposite direction.

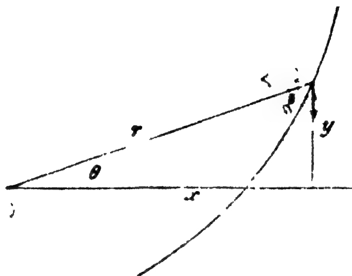
When there are two velocities or three velocities in two or in three rectangular directions, the resultant is the square root of the sum of their squares—and the cosines of the inclination of its direction to the given directions are the ratios of the components to the resultant.

11. *When a point moves in a plane curve, to express its component velocities at any instant along, and perpendicular*

to, the radius vector drawn from a fixed point in the plane of the curve.

Let x, y be its rectangular, r, θ its polar, co-ordinates; so that

$$x = r\cos\theta,\ \ y = r\sin\theta;$$



from which
$$\left.\begin{aligned}\frac{dx}{dt} &= \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\\ \frac{dy}{dt} &= \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\end{aligned}\right\} \cdots\cdots\cdots\cdots (1),$$
and

which are the velocities parallel to x and y. But by § 9 the velocity along the radius vector is

$$\frac{dy}{dt}\sin\theta + \frac{dx}{dt}\cos\theta = \frac{dr}{dt},\ \text{by (1)};$$

and the velocity perpendicular to it is

$$\frac{dy}{dt}\cos\theta - \frac{dx}{dt}\sin\theta = r\frac{d\theta}{dt},\ \text{by (1)}.$$

12. The velocity of a point is said to be accelerated or retarded according as it increases or diminishes, but the word *Acceleration* is generally used in both senses; and is defined as the rate of increase of the velocity per unit of time.

Acceleration may be either uniform or variable. It is said to be uniform when the point receives equal increments

of velocity in equal times, and is measured by the actual increase of velocity generated in unit of time. Let the unit of acceleration be so taken that a point under its action would receive an increment of a unit of velocity in a unit of time; then a point under the influence of α units of acceleration would receive an increment of α units of velocity in a unit of time, and consequently αt units of acceleration in t units of time. If the point starts from rest we have

$$v = \alpha t,$$

where v denotes the velocity at the end of the interval t, and α the acceleration.

13. Acceleration is variable when the point does not receive equal increments of velocity in equal increments of time. The acceleration at any instant is then measured by the increment of velocity which would have been generated in a unit of time had the acceleration remained constant during that interval and equal to the value at its commencement.

Let v be the velocity of the point at the end of the time t, α the acceleration at that instant, $v + \delta v$ the velocity at the end of the time $t + \delta t$; and let α_1, α_2 be the greatest and least values of the acceleration during the interval δt, then $\alpha_1 \delta t$, $\alpha_2 \delta t$ would be the increments of velocity in that interval, of a point under those accelerations respectively. But the actual acceleration is not greater than α_1 and not less than α_2, therefore the actual increment of velocity

δv is not greater than $\alpha_1 \delta t$ and not less than $\alpha_2 \delta t$,

or $\dfrac{\delta v}{\delta t}$ .... α_1 .... α_2,

however small δt may be. But, as δt continually diminishes, α_1 and α_2 tend continually to and ultimately become each equal to α. Therefore, proceeding to the limit,

$$\frac{dv}{dt} = \alpha.$$

The positive sign given to a shews that v increases with t, while a negative sign would shew that v decreases as t increases, in other words a negative acceleration is a retardation.

Combining the above equation with

$$\frac{ds}{dt} = v,$$

we have

$$\frac{d^2s}{dt^2} = a,$$

considering t as the independent variable.

If the point be in motion along a curve, the accelerations of the rates of increase of its co-ordinates are called the *Component Accelerations* of the point's velocity parallel to the axes. If these be denoted by a_x, a_y, a_z, we shall have

$$\frac{d^2x}{dt^2} = a_x, \quad \frac{d^2y}{dt^2} = a_y, \quad \frac{d^2z}{dt^2} = a_z.$$

With reference to these, $\sqrt{a_x^2 + a_y^2 + a_z^2}$ is called the *Resultant Acceleration.*

14. The acceleration $\frac{d^2s}{dt^2}$ is not the complete resultant of $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$, as may easily be seen: for its square does not equal the sum of the squares of those three accelerations, but it is the only part of their resultant which has any effect on the velocity; in short $\frac{d^2s}{dt^2}$ is the sum of the resolved parts of $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ in the direction of motion, as the following identical equation shews:

$$\frac{d^2s}{dt^2} = \frac{dx}{ds}\frac{d^2x}{dt^2} + \frac{dy}{ds}\frac{d^2y}{dt^2} + \frac{dz}{ds}\frac{d^2z}{dt^2}.$$

The other part of the resultant is at right angles to this, and shews its effect in changing the direction of the motion of the point. And this leads us to another form of acceleration, viz. when the velocity of the moving point is unaltered,

but the direction of motion changes. Its value will be given afterwards.

The above equation also shews, since $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ are the direction-cosines of the small arc ds which may have any direction whatever, that to obtain the acceleration along any line inclined at given angles to the axes, we must resolve the component accelerations parallel to the axes along it, and take the sum of the resolved parts. Thus the acceleration along a line inclined at angles λ, μ, ν to the axes is

$$\alpha_x \cos \lambda + \alpha_y \cos \mu + \alpha_z \cos \nu.$$

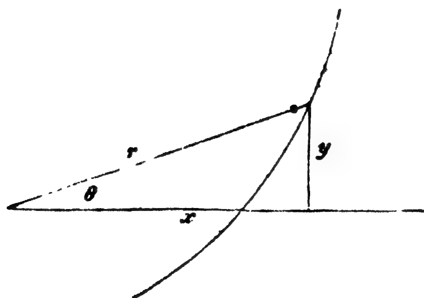
15. *When a point moves in a plane curve, to express its component accelerations at any instant along, and perpendicular to, the radius vector.*

Let x, y be the rectangular, r, θ the polar, co-cordinates; so that

$$x = r \cos \theta,$$

$$y = r \sin \theta;$$

we have
$$\frac{dx}{dt} = \frac{dr}{dt} \cos \theta - r \sin \theta \frac{d\theta}{dt},$$



and
$$\frac{d^2x}{dt^2} = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\} \cos \theta - \left(2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2}\right) \sin \theta.$$

Similarly,

$$\frac{d^2y}{dt^2} = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\} \sin\theta + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\cos\theta.$$

These are the accelerations parallel to x and y. Hence by § 14, the acceleration along the radius vector is

$$\frac{d^2y}{dt^2}\sin\theta + \frac{d^2x}{dt^2}\cos\theta,$$

$$\text{or}\quad \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2.$$

And the acceleration perpendicular to the radius vector is

$$\frac{d^2y}{dt^2}\cos\theta - \frac{d^2x}{dt^2}\sin\theta,$$

$$\text{or}\quad 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

which may be written $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$.

16. *When a point is in motion in any curve, to find its accelerations along, and perpendicular to, the tangent, at any instant.*

Let x, y, z be the co-ordinates of the point at the end of the time t, s the length of the arc described during that interval. Then, since by the equations to the curve x, y and z are functions of s,

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt};$$

$$\text{and}\quad \frac{d^2x}{dt^2} = \frac{d^2x}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{dx}{ds}\frac{d^2s}{dt^2}.$$

Similarly,

$$\frac{d^2y}{dt^2} = \frac{d^2y}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{dy}{ds}\frac{d^2s}{dt^2},$$

$$\frac{d^2z}{dt^2} = \frac{d^2z}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{dz}{ds}\frac{d^2s}{dt^2}.$$

To find the acceleration along the tangent, we must multiply these component accelerations by $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, respectively, and add. Thus the tangential acceleration is

$$\frac{dx}{ds}\frac{d^2x}{dt^2}+\frac{dy}{ds}\frac{d^2y}{dt^2}+\frac{dz}{ds}\frac{d^2z}{dt^2}=\frac{d^2s}{dt^2},$$

as we have already seen. Also in the normal, towards the center of curvature, we have the acceleration

$$\rho\left(\frac{d^2x}{ds^2}\frac{d^2x}{dt^2}+\ldots\ldots\right)=\rho\left(\frac{ds}{dt}\right)^2\left\{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2\right\}$$

$$=\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2.$$

We have assumed, in the above, the following equations from Analytical Geometry,

$$\frac{1}{\rho^2}=\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2,$$

where ρ is the radius of curvature;

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1,$$

$$\frac{dx}{ds}\frac{d^2x}{ds^2}+\frac{dy}{ds}\frac{d^2y}{ds^2}+\frac{dz}{ds}\frac{d^2z}{ds^2}=0.$$

17. We might have treated the component accelerations thus,

$$\left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2+\left(\frac{d^2z}{dt^2}\right)^2 \text{ or (resultant acceleration)}^2$$

$$=\frac{1}{\rho^2}\left(\frac{ds}{dt}\right)^4+\left(\frac{d^2s}{dt^2}\right)^2.$$

Now $\frac{d^2s}{dt^2}$ is the acceleration along the tangent, and the other part $\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2$, or $\frac{v^2}{\rho}$, acts at right angles to it as the

form of the equation shews, and consequently is the acceleration perpendicular to the tangent.

From the expressions for $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$, we also obtain

$$\frac{d^2x}{dt^2}\left(\frac{dy}{ds}\frac{d^2z}{ds^2}-\frac{dz}{ds}\frac{d^2y}{ds^2}\right)$$
$$+\frac{d^2y}{dt^2}\left(\frac{dz}{ds}\frac{d^2x}{ds^2}-\frac{dx}{ds}\frac{d^2z}{ds^2}\right)$$
$$+\frac{d^2z}{dt^2}\left(\frac{dx}{ds}\frac{d^2y}{ds^2}-\frac{dy}{ds}\frac{d^2x}{ds^2}\right)=0;$$

and thus the acceleration perpendicular to the osculating plane vanishes. The acceleration $\frac{v^2}{\rho}$ must therefore be along a normal to the path drawn in the osculating plane; that is, along the radius of absolute curvature.

18. We are therefore led to *expand* the definition given in § 12 thus:—Acceleration is the *rate of change of velocity whether that change take place in the direction of motion or not.*

What is meant by change of velocity is evident from § 10. For if a velocity *OA* become *OC*, its change is *AC*, or *OB*.

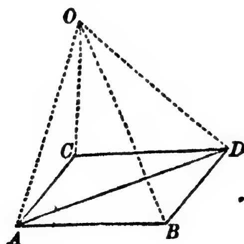
Hence, just as the direction of motion of a point is the tangent to its path—so the direction of acceleration of a moving point is to be found by the following construction.

From any point O draw lines OP, OQ, etc., representing in magnitude and direction the velocity of the moving point at every instant. The points, P, Q, etc., form in all cases of motion of a material particle a continuous curve, for an infinitely great force is requisite to change the velocity of a particle *abruptly* either in direction or magnitude. Now if Q be a point near to P, OP and OQ represent two successive values of the velocity. Hence PQ is the whole change of velocity during the interval. As the interval becomes smaller, the direction PQ more and more nearly becomes the tangent at P. Hence the direction of acceleration is that of the tangent to the curve thus described, called by its inventor, Sir W. R. Hamilton, the *Hodograph*.

The amount of acceleration is the rate of change of velocity, and is therefore measured by the velocity of P in the curve PQ.

19. The *Moment* of a velocity about any point is the rectangle under its magnitude and the perpendicular from the point upon its direction. The moment of the resultant velocity of a point about any point in the plane of the components is equal to the algebraic sum of the moments of the components, the proper sign of each moment depending on the *direction* of motion about the point. The same is true of moments of acceleration, and of momentum as defined later.

Consider two component velocities, AB and AC, and let AD be their resultant (§ 10). Their half moments round



the point O are respectively the areas OAB, OCA. Now OCA, together with half the area of the parallelogram $CABD$, is equal to OBD. Hence the sum of the two half moments

together with half the area of the parallelogram is equal to AOB together with BOD, that is to say, to the area of the whole figure $OABD$. But ABD, a part of this figure, is equal to half the area of the parallelogram; and therefore the remainder, OAD, is equal to the sum of the two half moments. And OAD is half the moment of the resultant velocity round the point O. Hence the moment of the resultant is equal to the sum of the moments of the two components. By attending to the *signs* of the moments, we see that the proposition holds when O is within the angle CAB.

20. Now if one of the components always passes through the point O, its moment vanishes. This is the case of a motion in which the acceleration is directed to a fixed point, and we thus prove the theorem that *in the case of acceleration always directed to a fixed point the path is plane and the areas described by the radius-vector are proportional to the times;* for the moment of velocity, which in this case is constant, is evidently double the rate at which the area is traced out by the radius-vector.

21. Hence in this case the velocity at any point is inversely as the perpendicular from the fixed point upon the tangent to the path, the momentary direction of motion.

For evidently the product of this perpendicular and the velocity at any instant gives double the area described in one second about the fixed point, which hast just been shewn to be a constant quantity.

22. The results of the last three sections may be easily obtained analytically, thus. Let the plane of motion be taken as that of x, y; and let the origin be the point about which moments are taken. Then if x, y be the position of the moving point at time t, the perpendicular from the origin on the tangent to its path is

$$p = x\frac{dy}{ds} - y\frac{dx}{ds} = r^2\frac{d\theta}{ds}, \text{ in polar co-ordinates.}$$

From this we have at once

$$p\frac{ds}{dt} = x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt}, \quad \ldots\ldots\ldots\ldots(1),$$

or with the notation of § 8,

$$pv = xv_y - yv_x,$$

which is the theorem of § 19.

$$\text{Also } \frac{d}{dt}(pv) = x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}, \quad\ldots\ldots\ldots\ldots\ (2).$$

Now, if the acceleration be directed to or from O, its moment about O which is evidently

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2},$$

must vanish. Hence (2) gives

$pv =$ constant; which is § 21.

By means of (1) this gives

$r^2\dfrac{d\theta}{dt} =$ constant, which is § 20;

since, if A be the area traced out by the radius-vector,

$$\frac{dA}{d\theta} = \frac{r^2}{2}.$$

23. *To determine the motion of a point when the accelerations to which it is subjected are given.*

This includes also, as will be seen, the determination of the motion when the component velocities are given.

Let α, β, γ be the given accelerations parallel to the axes, we have

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \alpha, \\ \frac{d^2y}{dt^2} &= \beta, \\ \frac{d^2z}{dt^2} &= \gamma, \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ (1).$$

Now α, β, γ may be functions of $x, y, z, t, \dfrac{dx}{dt}, \dfrac{dy}{dt}$, or $\dfrac{dz}{dt}$, or of two or more of these quantities. Equations (1) must

be integrated as simultaneous differential equations if possible. Thus we have the values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ in terms of one or more of the quantities x, y, z and t; that is the component velocities are known.

Another integration, if it can be performed, gives x, y, and z in terms of t; and, if the latter be eliminated from the three integrated equations, we have the two equations to the path in space, and thus theoretically the motion is completely determined.

It is unnecessary to give examples of the integration of such equations, as the major part of the following chapters will be devoted to them.

24. So far for a single point. When more points than one are considered, Kinematics enables us to determine, from the given motions of all, their *relative* motions with respect to any one of them; or conversely, from the actual motion of one, and the motions relative to it of the others, to determine the *actual* motions of the latter in space. This depends on the following self-evident proposition.

If the velocity of any point of a system be reversed in direction, and be communicated to each point of the system in composition with that which it already possesses, the relative motions of all about the first, thus reduced to rest, will be the same as their relative motions about it when all were in motion.

For the proof it is sufficient to notice that if at every instant the distance of two points, and the direction of the line joining them be the same as for two other points, the relative motions of one of each pair about the other will be the same. The simplest illustrations of this proposition are furnished by the relative motions of objects in a vessel or carriage, which are independent of the common velocity of the whole—or, on a grander scale, of terrestrial objects, whose relative motions are unaffected by the earth's rotation, or by its motion in space.

Since accelerations are compounded according to the same law as velocities, the above theorem is true of them also.

25. *Two points describe similar orbits about each other and about any point dividing in a given ratio the line which joins them.*

Let A and B be the points, G a point in AB such that $\frac{AG}{GB} =$ a constant.

The path of B about A will evidently be the same as that of A about B, since the length and direction of the



line AB ar

Also if G be fixed the path of B about it will evidently differ from that of B about A by having corresponding radii vectores diminished in the ratio $\frac{BG}{AB}$. But this is the definition of similar curves. The same of course would hold with respect to the relative path of A with respect to G. This proposition will be found of considerable use afterwards, as it enables us materially to simplify the equations of motion of two mutually attracting free particles.

26. As an instance of relative motion, *consider two points, one of which moves uniformly in a straight line, while the other moves uniformly in a circle about the first as center; to determine the path of the second point, the motion being in one plane.*

Take the line of motion of the first as the axis of x, v its velocity, the plane of the circle as xy, a the radius of the relative circular orbit, ω the angular velocity in it, § 32. Suppose the revolving point to be initially in the axis. Also at time t suppose the line joining the points to be inclined at an angle θ to the axis of x. Then for the co-ordinates of the revolving point we have

$$y = a \sin \theta,$$

$$x = vt + a \cos \theta.$$

But $\theta = \omega t$;

$$\text{hence } x = \frac{v}{\omega} \sin^{-1} \frac{y}{a} + \sqrt{(a^2 - y^2)}$$

is the equation to the absolute path required. This belongs to the class of cycloids; it is prolate or curtate according as v is greater or less than $a\omega$, or the absolute motion of the first point greater or less than that of the other in its circular orbit. If the two are equal, or $v = a\omega$, we have the equation to the common cycloid, as is indeed evident, for the circular path may be supposed the generating circle, and the velocity of the center in its rectilinear path is equal to that of the tracing point about that center.

27. It is evident that, whatever be the relative path, if r, θ denote the relative co-ordinates of the second point with respect to the first at time t, x, y, and $\bar{x}$ the absolute co-ordinates at the same time,

$$\left.\begin{array}{l} x = \bar{x} + r \cos\theta \\ y = r \sin\theta \end{array}\right\} \ldots\ldots (1).$$

Now in the first case, when the motion of the first point, and that in the relative orbit are given,

$\bar{x}$, r and θ are known functions of t, if therefore these values be substituted in (1), and t be eliminated, we shall have the equation between x and y, which is required.

Again, if the absolute orbits of both are given, x, y, and $\bar{x}$ are known in terms of t, and thus equations (1) serve to give r and θ in terms of t, which furnishes the complete determination of the relative path, and the circumstances of its description.

28. *In any system of moving points, to determine the relative, from the absolute, motions; and vice versâ.*

Let $x_1\, y_1\, z_1$, $x_2\, y_2\, z_2$ be the co-ordinates of two of the points, x, y, z the relative co-ordinates of the second with regard to the first, $u_1\, v_1\, w_1$, $u_2\, v_2\, w_2$ the velocities of each parallel to the axes, u, v, w the velocities of the second relatively to the first.

Then
$$\begin{aligned} x &= x_2 - x_1, & u &= u_2 - u_1, \\ y &= y_2 - y_1, & v &= v_2 - v_1, \\ z &= z_2 - z_1, & w &= w_2 - w_1. \end{aligned}$$

The second group may be derived from the first by differentiation with respect to t.

Now, when the *actual* motions of the two are given, all the subscribed quantities are known. Hence the above equations give the circumstances of the relative motion.

Or if the actual motion of the first, and the relative motion about it of the second, be known, we have $x\,y\,z$, $u\,v\,w$, $x_1\,y_1\,z_1$, $u_1\,v_1\,w_1$, to find the other six quantities for the actual motion of the second in space.

A second differentiation proves the statement in § 24 regarding relative acceleration.

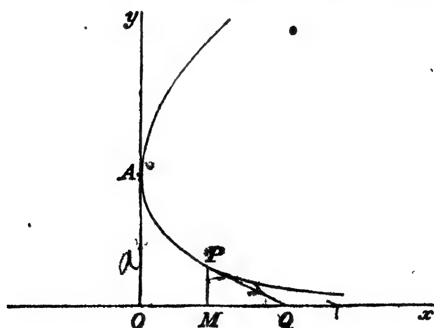
29. Some of the best illustrations of this part of our subject are to be found in what are called *Curves of Pursuit.*

These questions arose from the consideration of the path taken by a dog who in following his master always directs his course towards him.

In order to simplify the question the rates of motion of both master and dog are supposed to continue uniform; or at least to have a constant ratio.

30. As an instance of the curve of pursuit, suppose it be required *to determine the path of a point which continually with uniform velocity* u *moves towards another which is describing a straight line with uniform velocity* v.

The curve of course is plane. Take the line of motion



of the second point Q as the axis of x, and let $\bar{x}$ denote its position at the instant when the co-ordinates of the first, P, are x, y. The axis of y is chosen as that tangent to the curve of pursuit which is perpendicular to the axis of x, and the distance between the points in that position is a.

Let $\frac{v}{u} = e$, then

$$e\,AP = OQ, \text{ and } PQ \text{ is a tangent at } P.$$

These are our conditions, and lead to the following equations

$$es = \bar{x} = x - y\frac{dx}{dy}.$$

Differentiating with respect to y, we have

$$e\frac{ds}{dy} = -y\frac{d^2x}{dy^2}.$$

But s increases as y diminishes,

whence $$\frac{ds}{dy} = -\sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}.$$

Hence $$\frac{e}{y} = \frac{\frac{d^2x}{dy^2}}{\sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}},$$

and integrating, noting that $y = a$, $\frac{dx}{dy} = 0$ together,

$$\log\left(\frac{y}{a}\right)^e = \log\left[\sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}} + \frac{dx}{dy}\right].$$

Hence, $$\left(\frac{y}{a}\right)^e = \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}} + \frac{dx}{dy},$$

and therefore $\left(\frac{a}{y}\right)^e = \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}} - \frac{dx}{dy}$, taking reciprocals.

And we have finally

$$2\frac{dx}{dy} = \left(\frac{y}{a}\right)^e - \left(\frac{a}{y}\right)^e \text{ .... (1),}$$

$$\text{or } 2(x + C) = \frac{y^{e+1}}{a^e(e+1)} + \frac{a^e}{y^{e-1}(e-1)}.$$

But $x = 0$, $y = a$; which gives $C = \frac{ae}{e^2 - 1}$.

$$\text{Hence } 2\left(x + \frac{ae}{e^2 - 1}\right) = \frac{y^{e+1}}{a^e(e+1)} + \frac{a^e}{y^{e-1}(e-1)} \text{.........(2).}$$

This is true for all values of e except unity, a case to which we will presently recur.

There are two cases of curves represented by equation (2). 1st, $e > 1$, 2nd, $e < 1$.

In the first case Q moves the faster, and P can never overtake it; the curve therefore never meets the axis of x, which indeed will be seen by (2) to be an asymptote.

In the second case equation (2) becomes

$$2\left(x - \frac{ae}{1 - e^2}\right) = \frac{y^{1+e}}{a^e(1+e)} - \frac{a^e y^{1-e}}{1-e},$$

and for $x = \frac{ae}{1 - e^2}$ we have $y = 0$, and also by (1) $\frac{dx}{dy}$ infinite. Hence the curve touches the axis at this point. The remainder of the curve satisfies a modified form of the question, and is called the *Curve of Flight*. {It is to be observed, however, that $x = \frac{ae}{1 - e^2}$ gives also $y = \pm a\left(\frac{1+e}{1-e}\right)^{\frac{1}{2e}}$}.

When $e = 1$, the integral of (1) is

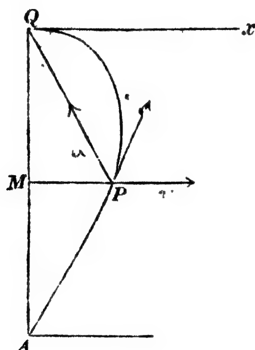
$$2\left(x + \frac{a}{4}\right) = \frac{y^2}{2a} - a \log\frac{y}{a};$$

the only case in which we do not obtain an algebraic curve. Here again the axis of x is an asymptote.

31. As an instance of relative motion let us consider the path of P with regard to Q. It will be easy to see that this corresponds exactly to the following question.

A boat, propelled (relatively to the water) with uniform velocity u, *starts from a point* A *in the bank of a river which runs with velocity* v *parallel to* Qx, *and tends continually to the point* Q, *on the other bank, directly opposite to* A; *to find its path.*

The constant velocity of the stream in this case communicated to P corresponds to the constant velocity of Q in the last example, but is in the opposite direction. In fact, if the earth were to be supposed moving in the direction xQ with uniform velocity v, the river would be at rest in space, and the *actual* motions of P and Q would be the same as in the last example. (See § 24.)



To investigate the path, take Q as origin, Qx, QA as the axes. Then the component velocities of P are v parallel to Qx and u along PQ, and the tangent to its path is in the direction of the resultant of these two. Putting θ for PQx,

we have $$\frac{dy}{dx} = -\frac{u \sin\theta}{v - u\cos\theta}$$

$$= -\frac{\sin\theta}{e - \cos\theta}$$

$$= -\frac{y}{e\sqrt{(x^2+y^2)} - x}.$$

This, being a homogeneous equation, is easily integrated, and we have, taking $x=0$, $y=a$ together,

$$\frac{y^{1+e}}{a^e} = \sqrt{(x^2+y^2)} - x \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

$$\text{or} \quad \left(\frac{\rho \sin\theta}{a}\right)^e = \frac{1-\cos\theta}{\sin\theta},$$

in polar co-ordinates. This evidently gives a parabola about Q as focus, if $e=1$.

To find the time of crossing the stream.

This may easily be effected by considering the actual velocity parallel to the axis of y,

$$\frac{dy}{dt} = -u\sin\theta$$

$$= -u\frac{y}{\sqrt{(x^2+y^2)}}.$$

Now taking quotients of y^2 by both sides of (1),

$$a^e y^{1-e} = \sqrt{(x^2+y^2)} + x.$$

Hence $\qquad 2\sqrt{(x^2+y^2)} = a^e y^{1-e} + a^{-e}y^{1+e};$

and therefore $\qquad \frac{dy}{y}(a^e y^{1-e} + a^{-e}y^{1+e}) = -2u\,dt.$

Taking the integral from a to 0, and putting T_1 for the time of crossing,

$$\frac{a}{1-e^2} = uT_1; \quad \text{or} \quad T_1 = \frac{au}{u^2-v^2}.$$

But, if there had been no current, we should have had for the time of crossing,

$$T_0 = \frac{a}{u}; \quad \text{whence} \quad \frac{T_1}{T_0} = \frac{u^2}{u^2-v^2}.$$

In the integration we have, of course, $e<1$, else the boat could not reach Q.

If $e=1$, the boat will reach the farther bank but not at Q. The solution of this case presents no special difficulty.

32. If the motion of a point in a plane be considered with reference to a fixed point in that plane, the rate of increase of the angle made by the line joining the two, with some fixed line in the plane, is called the *Angular Velocity* of the former point about the latter.

Suppose this angle to be represented by θ at time t; then at time $t+\delta t$ it has the value $\theta+\delta\theta$, and it may be shewn as before (§ 7), that if ω represent the angular velocity required, then

$$\omega=\frac{d\theta}{dt}.$$

Ex., *A point moves uniformly, with velocity* v, *in a straight line; to find at any instant its angular velocity about a fixed point whose distance from the straight line is* a.

Taking as initial line the perpendicular from the fixed point on the line of motion; the polar equation of the path is

$$r=a\sec\theta.$$

Also, if when $t=0$, $\theta=0$, we have

$$r\sin\theta=vt.$$

Hence, $a\tan\theta=vt$,

$$\text{and}\quad \omega=\frac{d\theta}{dt}=\frac{va}{a^2+v^2t^2}=\frac{va}{r^2}.$$

33. *A point describes a circle with uniform velocity; it is required to find the actual velocity, and the angular velocity (about the center) in any orthographic projection.*

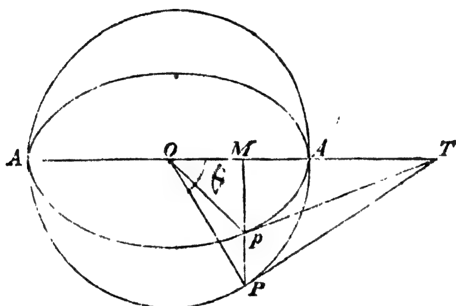
Let ApA' be an ellipse and APA' the auxiliary circle. Then the former will be the orthographic projection if its axes be made in the ratio of the cosine of the angle (α) between the planes of projection. Also if PpM be perpendicular to AA', P and p will be corresponding points in the two. Draw the tangents pT, PT; then

$$\frac{\text{actual velocity at } p}{\text{.................... } P} = \frac{pT}{PT}, \text{ and if } TOP = \theta,$$

$$\frac{\text{velocity at } p}{\text{............ } P} = \frac{\sqrt{(PT^2 \sin^2 \theta + PT^2 \cos^2 \theta \cos^2 \alpha)}}{PT}$$

$$= \sqrt{(\sin^2 \theta + \cos^2 \theta \cos^2 \alpha)}$$

$$= \sqrt{(1 - \sin^2 \alpha \cos^2 \theta)}.$$



Now, if $TOp = \phi$, $\dfrac{\text{angular velocity at } p}{\text{...................... } P} = \dfrac{d\phi}{d\theta}$

$$= \frac{d}{d\theta} \tan^{-1} (\cos \alpha \tan \theta)$$

$$= \frac{\cos \alpha}{\cos^2 \theta + \cos^2 \alpha \sin^2 \theta}$$

$$= \frac{\cos \alpha}{1 - \sin^2 \alpha \sin^2 \theta}.$$

This is a maximum if $\theta = \dfrac{\pi}{2}$, when its value is $\sec \alpha$,

.... minimum ...... $= 0$ .... $\cos \alpha$.

Hence, if ω_1 and ω_2 be the greatest and least angular velocities in the projection,

$\sqrt{(\omega_1 \omega_2)}$ is the angular velocity in the original path.

34. Evidently, the product of the radius-vector into the angular velocity is the velocity perpendicular to the radius-vector. This is to the whole velocity as the perpendicular on the tangent is to the radius-vector; and therefore the product of the square of the radius-vector by the angular velocity is equal to the product of the whole velocity by the perpendicular on the tangent, *i.e.* to the moment of velocity about the pole, § 22, (1).

35. The rate of increase or diminution of the angular velocity when variable is called the *Angular Acceleration*, and is measured with reference to the same unit angle.

36. *The motion of a point in a plane being given with respect to fixed axes, to investigate expressions for its velocity and acceleration relative to axes in the same plane, which revolve about a common origin with uniform angular velocity.*

Let ω be this angular velocity, then, if at time $t=0$ the fixed and revolving axes coincide, at time t they will be inclined at an angle ωt. Hence, if x, y, ξ, η be the co-ordinates of the point at time t, referred to the fixed and the revolving axes respectively, we have

$$\left.\begin{aligned}\xi &= x\cos\omega t + y\sin\omega t\\ \eta &= y\cos\omega t - x\sin\omega t\end{aligned}\right\} \quad\cdots\cdots\cdots\cdots\cdots\cdots \quad (1).$$

These give, by differentiation,

$$\frac{d\xi}{dt} = \frac{dx}{dt}\cos\omega t + \frac{dy}{dt}\sin\omega t - \omega\,(x\sin\omega t - y\cos\omega t)$$

$$\left.\begin{aligned} &= \frac{dx}{dt}\cos\omega t + \frac{dy}{dt}\sin\omega t + \omega\eta.\\ \text{Similarly, } \frac{d\eta}{dt} &= \frac{dy}{dt}\cos\omega t - \frac{dx}{dt}\sin\omega t - \omega\xi\end{aligned}\right\} \quad\cdots\cdots\quad (2),$$

which determine the relative velocities.

Again,

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2} &= \frac{d^2x}{dt^2}\cos\omega t + \frac{d^2y}{dt^2}\sin\omega t - 2\omega\left(\frac{dx}{dt}\sin\omega t - \frac{dy}{dt}\cos\omega t\right) - \omega^2\xi\\ \frac{d^2\eta}{dt^2} &= \frac{d^2y}{dt^2}\cos\omega t - \frac{d^2x}{dt^2}\sin\omega t - 2\omega\left(\frac{dy}{dt}\sin\omega t + \frac{dx}{dt}\cos\omega t\right) - \omega^2\eta\end{aligned}\right\} \quad (3)$$

or

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2} &= \frac{d^2x}{dt^2}\cos\omega t + \frac{d^2y}{dt^2}\sin\omega t + 2\omega\frac{d\eta}{dt} + \omega^2\xi\\ \frac{d^2\eta}{dt^2} &= \frac{d^2y}{dt^2}\cos\omega t - \frac{d^2x}{dt^2}\sin\omega t - 2\omega\frac{d\xi}{dt} + \omega^2\eta\end{aligned}\right\}\ldots\ldots(3'),$$

the relative accelerations.

Now the component accelerations along *fixed* axes, which at the time t coincide with the moving axes, are evidently represented by the first two terms of the right-hand sides of these equations; or, in terms of the co-ordinates with respect to the moving axes, by

$$\frac{d^2\xi}{dt^2} - 2\omega\frac{d\eta}{dt} - \omega^2\xi, \quad \text{and} \quad \frac{d^2\eta}{dt^2} + 2\omega\frac{d\xi}{dt} - \omega^2\eta\ldots\ldots(4).$$

Ex. If the point be at rest, x and y are constant, and

$$\frac{d\xi}{dt} = \omega\eta, \qquad \frac{d\eta}{dt} = -\omega\xi.$$

Also
$$\frac{d^2\xi}{dt^2} = -\omega^2\xi, \quad \frac{d^2\eta}{dt^2} = -\omega^2\eta.$$

These expressions are obvious, as in this case the relative motion of the point with respect to the moving axes is a uniform circular motion about the origin, in the *negative* direction, *i. e.* from the axis of η to that of ξ.

37. *Suppose the axes not to revolve uniformly.*

In this case the investigation is precisely the same as the above, with the exception that θ, a given function of t, must be substituted for ωt. If ω, now no longer constant, be put for $\frac{d\theta}{dt}$, the student will have no difficulty in verifying the following expressions which take the place of (2), (3′) and (4), of the preceding section.

$$\left.\begin{aligned}\frac{d\xi}{dt}&=\frac{dx}{dt}\cos\theta+\frac{dy}{dt}\sin\theta+\omega\eta\\ \frac{d\eta}{dt}&=\frac{dy}{dt}\cos\theta-\frac{dx}{dt}\sin\theta-\omega\xi\end{aligned}\right\}\ldots\ldots\ldots\ldots(2_1).$$

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2}&=\frac{d^2x}{dt^2}\cos\theta+\frac{d^2y}{dt^2}\sin\theta+\omega^2\xi+2\omega\frac{d\eta}{dt}+\frac{d\omega}{dt}\eta\\ \frac{d^2\eta}{dt^2}&=\frac{d^2y}{dt^2}\cos\theta-\frac{d^2x}{dt^2}\sin\theta+\omega^2\eta-2\omega\frac{d\xi}{dt}-\frac{d\omega}{dt}\xi\end{aligned}\right\}\ldots(3'_1).$$

$$\frac{d^2\xi}{dt^2}-\omega^2\xi-\frac{1}{\eta}\frac{d}{dt}(\omega\eta^2),\quad \frac{d^2\eta}{dt^2}-\omega^2\eta+\frac{1}{\xi}\frac{d}{dt}(\omega\xi^2)\ldots\ldots\ldots(4_1).$$

These expressions might have been deduced at once from the expressions in § 15, by the consideration of relative accelerations as in § 24. Let $OM=\xi$, $MP=\eta$, be the co-ordinates of the point referred to the moving axes. Then, by § 15, the acceleration of M along OM is

$$\frac{d^2\xi}{dt^2}-\omega^2\xi.$$

Also, as MP revolves with angular velocity ω, the acceleration of P relative to M in the direction perpendicular to MP, is

$$\frac{1}{\eta}\frac{d}{dt}(\omega\eta^2).$$

This is in the direction of the negative part of the axis of ξ. Hence the resolved part parallel to $O\xi$, of the relative acceleration of P with respect to O, is

$$\frac{d^2\xi}{dt^2}-\omega^2\xi-\frac{1}{\eta}\frac{d}{dt}(\omega\eta^2).$$

38. The principles already enunciated, and the examples given of their application, will suffice for the solution of problems on this part of the subject.

Other examples of the application of these principles,

such as the kinematical part of the investigations of the Hodograph, Tractory, &c., will be more appropriately introduced in future chapters.

EXAMPLES.

(1) A point moves from rest in a given path, and its velocity at any instant is proportional to the time elapsed since its motion commenced; find the space described in a given time.

(2) If a point begin to move with velocity v, and at equal intervals of time a velocity u be communicated to it in the same direction; find the space described in n such intervals.

(3) A man six feet high walks in a straight line at the rate of four miles an hour away from a street lamp, the height of which is 10 feet; supposing the man to start from the lamp-post, find the rate at which the end of his shadow travels, and also the rate at which the end of his shadow separates from himself.

(4) If the position of a point moving in a plane be determined by the co-ordinates ρ and ϕ, ρ being measured from a fixed circle (radius a) along a tangent which has revolved through an angle ϕ from a fixed tangent; investigate the following expressions for the accelerations along and perpendicular to ρ respectively,

$$\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2 + a\frac{d^2\phi}{dt^2}$$

$$\text{and}\quad \frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\phi}{dt}\right) + a\left(\frac{d\phi}{dt}\right)^2.$$

(5) Prove that it is not possible for a point to move so that its velocity at any point may be proportional to the length of the path which it has described from rest: also that if its velocity be proportional to the space it has to describe, however small, it will never accomplish it.

(6) The velocity of a point parallel to each of three rectangular axes is proportional to the product of the other two co-ordinates; what are the equations to the path, and what is the time of describing a given portion when the curve passes through the origin?

(7) A point moves in a plane, its velocities parallel to the axes of x and y are

$$u + ey \text{ and } v + \epsilon x \text{ respectively,}$$

shew that it moves in a conic section.

(8) Two points are moving with uniform velocity in two straight lines, 1st in a plane, 2nd in space; given the initial circumstances, find when they are nearest to each other. Shew also that in both cases the relative path is a straight line, described with uniform velocity.

(9) A number of points are moving with uniform velocity in straight lines in space; determine the motion of their common center of inertia. (§ 53.)

(10) A cannon-ball is moving in a direction making an acute angle θ with a line drawn from the ball to an observer; if V be the velocity of sound, and nV that of the ball, prove that the whizzing of the ball at different points of its course will be heard in the order in which it is produced, or in the reverse order, according as $n <> \sec\theta$.

(11) A particle projected with a velocity u, is acted on by a force, which produces a constant acceleration f, in the plane of motion, inclined at a constant angle α to the direction of motion. Obtain the intrinsic equation to the curve described, and shew that the particle will be moving in the opposite direction to that of projection at the time

$$\frac{u}{f\cos\alpha}\left(e^{\pi\cot\alpha} - 1\right).$$

(12) Shew that any infinitely small motion given to a plane figure in its own plane is equivalent to a rotation through an infinitely small angle about some point in the figure.

(13) The highest point of the wheel of a carriage rolling on a road moves twice as fast as each of two points in the rim whose distance from the ground is half the radius of the wheel.

(14) A rod of given length moves with its extremities in two given lines which intersect; shew how to draw a tangent to the path described by any point of the rod.

(15) Investigate the position of the instantaneous center about which the rod is turning, and apply this also to solve the preceding question.

(16) One circle rolls on another whose center is fixed. From the initial and final positions of a diameter in each determine how much of their circumferences have been in contact.

(17) One point describes the diameter AB of a circle with uniform velocity, and another the semi-circumference AB from rest with uniform tangential acceleration, they start together from A and arrive together at B, shew that the velocities at B are as $\pi : 1$.

(18) In the example of § 30 find in the case of $e = 1$ the ultimate distance of the particles, and for $e < 1$ the length of time occupied in the pursuit.

(19) In the example of § 31 find the greatest distance the boat is carried down the stream, and shew that when it is in that position its velocity is $\sqrt{(u^2 - v^2)}$.

When $u = v$, shew directly that the curve described is a parabola.

(20) Shew that if ρ be the radius of curvature of the curve of pursuit, we have in the figure of § 30,

$$\rho = \frac{PQ^2}{ePM}.$$

(21) In the case of a boat propelled with velocity u relatively to the water in a stream running with velocity v; shew that the boat passes from one given point to another in the least possible time when its actual path is a straight line.

(22) The velocity of a stream varies as the distance from the nearest bank; shew that a man attempting to swim directly across will describe two semiparabolas. (Shew that the sub-normal is constant.) Find by how much the mean velocity is increased.

(23) A point moves uniformly in a circle; find an expression for its angular velocity about any point in the plane of the circle.

(24) If the velocity of a point moving in a plane curve vary as the radius of curvature, shew that the direction of motion revolves with uniform angular velocity.

(25) Two bevilled wheels roll together; having given the angular velocity of the first wheel and the inclinations of the axes of the cones, find their vertical angles that the second may revolve with given angular velocity.

(26) Supposing the Earth and Venus to describe in the same plane circles about the Sun as center; investigate an expression for the angular velocity of the Earth about Venus in any position, the actual velocities being inversely as the square roots of their distances from the Sun.

(27) A particle moving uniformly round the circular base of an oblique cone is projected by generating lines on a sub-contrary section; find its angular velocity about the center of the latter.

(28) If ξ, η denote the co-ordinates of a moving point referred to two axes, one of which is fixed and the other rotates with uniform angular velocity ω, prove that its component accelerations parallel to these axes are

$$\frac{d^2\xi}{dt^2} - 2\omega \operatorname{cosec} \omega t \frac{d\eta}{dt},$$

$$\frac{d^2\eta}{dt^2} - \omega^2\eta + 2\omega \cot \omega t \frac{d\eta}{dt}.$$

CHAPTER II.

LAWS OF MOTION.

39. HAVING, in the preceding chapter, considered the purely geometrical properties of the motion of a point or particle, we must now treat of the causes which produce various circumstances of motion; and of the experimental laws, on the assumed truth of which all our succeeding investigations are founded. And it is obvious that we now introduce for the first time the idea of *Matter*.

We commence with a few definitions and explanations, necessary to the full enunciation of Newton's Laws and their consequences.

40. The *Quantity of Matter* in a body, or the *Mass* of a body, is proportional to the *Volume* and the *Density* conjointly. The *Density* may therefore be defined as the quantity of matter in unit volume.

If M be the mass, ρ the density, and V the volume, of a homogeneous body, we have at once

$$M = V\rho;$$

if we so take our units that unit of mass is the mass of unit volume of a body of unit density. If the density vary from point to point, we have, of course,

$$M = \iiint \rho\, dV.$$

As will be presently explained, the most convenient unit mass is an imperial pound of matter.

41. A *Particle* of matter is supposed to be so small that, though retaining its material properties, it may be treated so

far as its co-ordinates, &c. are concerned, as a geometrical point.

42. The *Quantity of Motion*, or the *Momentum*, of a moving body is proportional to its mass and velocity conjointly.

Hence, if we take as unit of momentum the momentum of a unit of mass moving with unit velocity, the momentum of a mass M moving with velocity v is Mv.

43. *Change of Quantity of Motion*, or *Change of Momentum*, is proportional to the mass moving and the change of its velocity conjointly.

Change of velocity is to be understood in the general sense of § 10. Thus, with the notation of that section, if a velocity represented by OA be changed to another represented by OC, the change of velocity is represented in magnitude and direction by AC.

44. *Rate of Change of Momentum*, or *Acceleration of Momentum*, is proportional to the mass moving and the acceleration of its velocity conjointly. Thus (§ 16) the acceleration of momentum of a particle moving in a curve is $M\frac{d^2s}{dt^2}$ along the tangent, and $M\frac{v^2}{\rho}$ in the radius of absolute curvature.

45. The *Vis Viva*, or *Kinetic Energy*, of a moving body is proportional to the mass and the square of the velocity, conjointly. If we adopt the same units of mass and velocity as before, there is particular advantage in defining kinetic energy as *half* the product of the mass into the square of its velocity.

46. *Rate of Change of Kinetic Energy* (when defined as above) is the product of the velocity into the component of acceleration of momentum in the direction of motion.

$$\text{For } \frac{d}{dt}\left(\frac{Mv^2}{2}\right) = Mv\frac{dv}{dt} = v\left(M\frac{d^2s}{dt^2}\right).$$

47. Matter has an innate power of resisting external influences, so that every body, as far as it can, remains at rest, or moves uniformly in a straight line.

This, the *Inertia* of matter, is proportional to the quantity of matter in the body. And it follows that some *cause* is requisite to disturb a body's uniformity of motion, or to change its direction from the natural rectilinear path.

48. *Impressed Force*, or *Force* simply, is any cause which tends to alter a body's natural state of rest, or of uniform motion in a straight line.

The three elements specifying a force, or the three elements which must be known, before a clear notion of the force under consideration can be formed, are, its place of application, its direction, and its magnitude.

49. The *Measure of a Force* is the quantity of motion which it produces in unit of time. According to this method of measurement, *the standard or unit force is that force which, acting on the unit of matter during the unit of time, generates the unit of velocity.*

Hence the British absolute unit force is the force which, acting on one pound of matter for one second, generates a velocity of one foot per second.

[According to the common system followed in modern mathematical treatises on dynamics, the unit of mass is g times the mass of the standard or unit weight; g being the numerical value of the acceleration produced (in some particular locality) by the earth's attraction on falling bodies. This definition, giving a varying and a very unnatural unit of mass, is exceedingly inconvenient. In reality, standards of weight are *masses*, not *forces*. They are employed primarily in commerce for the purpose of measuring out a definite *quantity* of matter; not an amount of matter which shall be attracted by the earth with a given force.]

50. To render this standard intelligible, all that has to be done is to find how many absolute units will produce, in any particular locality, the same effect as the force of gravity.

The way to do this is to measure the effect of gravity in producing acceleration on a body unresisted in any way. The most accurate method is indirect, by means of the pendulum. The result of pendulum experiments made at Leith Fort, by Captain Kater, is, that the velocity acquired by a body falling unresisted for one second is at that place 32·207 feet per second. The variation in the force of gravity for one degree of difference of latitude about the latitude of Leith is only ·0000832 of its own amount. The average value for the whole of Great Britain differs but little from 32·2; that is, the force of gravity on a pound of matter in this country is 32·2 times the force which, acting on a pound for a second, would generate a velocity of one foot per second; in other words, 32·2 is the number of absolute units which measures the weight of a pound. Thus, speaking very roughly, the British absolute unit of force is equal to the weight of about half an ounce.

51. Forces (since they involve only direction and magnitude) may be represented, as velocities are, by straight lines in their directions, and of lengths proportional to their magnitudes, respectively.

Also the laws of composition and resolution of any number of forces acting at the same point, are, as we shall presently shew, § 62, the same as those which we have already proved to hold for velocities; so that, with the substitution of force for velocity, § 10 is still true.

52. The *Component* of a force in any direction, sometimes called the *Effective Component* in that direction, is therefore found by multiplying the magnitude of the force by the cosine of the angle between the directions of the force and the component. The remaining component in this case is perpendicular to the other.

It is very generally convenient to resolve forces into components parallel to three lines at right angles to each other; each such resolution being effected by multiplying by the cosine of the angle concerned.

The magnitude of the resultant of two, or of three, forces

in directions at right angles to each other, is the square root of the sum of their squares.

53. The *Center of Inertia or Mass* of any system of material points whatever (whether rigidly connected with one another, or connected in any way, or quite detached), is a point whose distance from any plane is equal to the sum of the products of each mass into its distance from the same plane divided by the sum of the masses.

The distance from the plane of yz, of the center of inertia of masses m_1, m_2, etc., whose distances from the plane are x_1, x_2, etc., is therefore

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \text{etc.}}{m_1 + m_2 + \text{etc.}} = \frac{\Sigma (mx)}{\Sigma m}.$$

And, similarly, for the other co-ordinates.

Hence its distance from the plane

$$\delta = \lambda x + \mu y + \nu z - a = 0,$$

is
$$D = \lambda \bar{x} + \mu \bar{y} + \nu \bar{z} - a,$$

$$= \frac{\Sigma \{m(\lambda x + \mu y + \nu z - a)\}}{\Sigma m} = \frac{\Sigma (m\delta)}{\Sigma m},$$

as stated above. And its velocity perpendicular to that plane is

$$\frac{dD}{dt} = \frac{1}{\Sigma m} \Sigma \left\{ m \left(\lambda \frac{dx}{dt} + \mu \frac{dy}{dt} + \nu \frac{dz}{dt} \right) \right\} = \frac{\Sigma \left(m \frac{d\delta}{dt} \right)}{\Sigma m},$$

from which, by multiplying by Σm, and noting that δ is the distance of x, y, z from $\delta = 0$, we see that the sum of the momenta of the parts of the system in any direction is equal to the momentum in that direction of the whole mass collected at the center of inertia.

54. By introducing, in the definition of moment of velo-

city (§ 19), the mass of the moving body as a factor, we have an important element of dynamical science, the *Moment of Momentum.* The laws of composition and resolution are the same as those already explained.

55. A force is said to *do Work* if it moves the body to which it is applied, and the work done is measured by the resistance overcome, and the space through which it is overcome, conjointly.

Thus, in lifting coals from a pit, the amount of work done is proportional to the weight of the coals lifted; that is, to the force overcome in raising them; and also to the height through which they are raised. The unit for the measurement of work adopted in practice by British engineers, is that required to overcome the weight of a pound through the space of a foot, and is called a foot-pound.

In purely scientific measurements, the unit of work is not the foot-pound, but the kinetic unit force (§ 49) acting through unit of space.

If the weight be raised obliquely, as, for instance, along a smooth inclined plane, the space through which the force has to be overcome is increased in the ratio of the length to the height of the plane; but the force to be overcome is not the whole weight, but only the resolved part of the weight parallel to the plane; and this is less than the weight in the ratio of the height of the plane to its length. By multiplying these two expressions together, we find, as we might expect, that the amount of work required is unchanged by the substitution of the oblique for the vertical path.

56. Generally, if s be an arc of the path of a particle, S the tangential component of the applied forces, the work done on the particle between any two points of its path is

$$\int S ds,$$

taken between limits corresponding to the initial and final positions.

Referred to rectangular co-ordinates, it is easy to see, by the law of resolution of forces, § 62, that this becomes

$$\int\left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right)ds.$$

Thus it appears that, for any force, the work done during an indefinitely small displacement of the point of application is the product of the resolved part of the force in the direction of the displacement into the displacement.

From this it follows, that if the motion of a body be always perpendicular to the direction in which a force acts, such a force does no work. Thus the mutual normal pressure between a fixed and a moving body, the tension of the cord to which a pendulum bob is attached, the attraction of the sun on a planet if the planet describe a circle with the sun in the center, are all cases in which no work is done by the force.

In fact the geometrical condition that the resultant of X, Y, Z, shall be perpendicular to ds is

$$X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}=0,$$

and this makes the above expression for the work vanish.

57. Work done on a body by a force is always shewn by a corresponding increase of vis viva, or kinetic energy, if no other forces act on the body which can do work or have work done against them. If work be done against any forces, the increase of kinetic energy is less than in the former case by the amount of work so done. In virtue of this, however, the body possesses an equivalent in the form of *Potential Energy*, if its physical conditions are such that these forces will act equally, and in the same directions, if the motion of the system is reversed. Thus there may be no change of kinetic energy produced, and the work done may be wholly stored up as potential energy.

Thus a weight requires work to raise it to a height, a spring requires work to bend it, air requires work to com-

press it, etc.; but a raised weight, a bent spring, compressed air, etc., are *stores* of energy which can be made use of at pleasure.

These definitions being premised, we give Newton's *Laws of Motion.*

58. LAW I. *Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by impressed forces to change that state.*

We may logically convert the assertion of the first law of motion as to velocity into the following statements:—

The times during which any particular body, not compelled by force to alter the speed of its motion, passes through equal spaces, are equal. And, again—Every other body in the universe, not compelled by force to alter the speed of its motion, moves over equal spaces in successive intervals, during which the particular chosen body moves over equal spaces.

59. The first part merely expresses the convention universally adopted for the measurement of *Time.* The earth, in its rotation about its axis, presents us with a case of motion in which the condition of not being compelled by force to alter its speed, is more nearly fulfilled than in any other which we can easily or accurately observe. Hence the numerical measurement of time practically rests on defining *equal intervals of time,* as *times during which the earth turns through equal angles.* This is, of course, a mere convention, and not a law of nature; and, as we now see it, is a part of Newton's first law.

The remainder of the law is not a convention, but a great truth of nature, which we may illustrate by referring to small and trivial cases as well as to the grandest phenomena we can conceive.

60. LAW II. *Change of motion is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.*

We have considered change of velocity, or acceleration,

as a purely geometrical quantity, and have seen how it may be at once inferred from the given initial and final velocities of a body. By the definition of motion, or quantity of motion (§ 42), we see that, if we multiply the change of velocity, thus geometrically determined, by the mass of the body, we have the change of motion (§ 43) referred to in Newton's law as the measure of the force which produces it.

It is to be particularly noticed, that in this statement there is nothing said about the actual motion of the body before it was acted on by the force: it is only the *change* of motion that concerns us. Thus the same force will produce precisely the same change of motion in a body, whether the body be at rest, or in motion with any velocity whatever.

61. Again, it is to be noticed that nothing is said as to the body being under the action of *one* force only; so that we may logically put part of the second law in the following (apparently) amplified form:—

When any forces whatever act on a body, then, whether the body be originally at rest or moving with any velocity and in any direction, each force produces in the body the exact change of motion which it would have produced if it had acted singly on the body originally at rest.

62. A remarkable consequence follows immediately from this view of the second law. Since forces are measured by the changes of motion they produce, and their directions assigned by the directions in which these changes are produced; and since the changes of motion of one and the same body are in the directions of, and proportional to, the changes of velocity—a single force, measured by the resultant change of velocity, and in its direction, will be the equivalent of any number of simultaneously acting forces. Hence

The resultant of any number of forces (applied at one point) is to be found by the same geometrical process as the resultant of any number of simultaneous velocities.

From this follows at once (§ 10) the construction of the *Parallelogram of Forces* for finding the resultant of two

forces acting at the same point, and the *Polygon of Forces* for the resultant of any number of forces acting at a point. And, so far as a single particle is concerned, we have at once the whole subject of Statics.

63. The second law gives us the means of measuring force, and also of measuring the mass of a body.

For, if we consider the actions of various forces upon the same body for equal times, we evidently have changes of velocity produced, which are *proportional to* the forces. The changes of velocity, then, give us in this case the means of comparing the magnitudes of different forces. Thus the velocities acquired in one second by the same mass (falling freely) at different parts of the earth's surface, give us the relative amounts of the earth's attraction at these places.

Again, if equal forces be exerted on different bodies, the changes of velocity produced in equal times must be *inversely* as the masses of the various bodies. This is approximately the case, for instance, with trains of various lengths drawn by the same locomotive.

Again, if we find a case in which different bodies, each acted on by a force, acquire in the same time the same changes of velocity, the forces must be proportional to the masses of the bodies. This, when the resistance of the air is removed, is the case of falling bodies; and from it we conclude that *the weight of a body in any given locality, or the force with which the earth attracts it, is proportional to its mass.*

64. It appears, lastly, from this law, that every theorem of Kinematics connected with acceleration has its counterpart in Kinetics. Thus, for instance (§ 16), we see that the force, under which a particle describes any curve, may be resolved into two components, one in the tangent to the curve, the other *towards* the center of curvature; their magnitudes being the acceleration of momentum, and the product of the momentum into the angular velocity about the center of curvature, respectively. In the case of uniform motion, the first of these vanishes, or the whole force is perpendicular to the direction of motion. When there is

no force perpendicular to the direction of motion, there is no curvature, or the path is a straight line.

Hence, if we resolve the forces, acting on a particle of mass m whose co-ordinates are x, y, z, into the three rectangular components X, Y, Z; we have

$$m\frac{d^2x}{dt^2}=X,\quad m\frac{d^2y}{dt^2}=Y,\quad m\frac{d^2z}{dt^2}=Z.$$

In many of the future chapters these equations will be somewhat simplified by assuming unity as the mass of the moving particle. When this cannot be done, it is sometimes convenient to assume X, Y, Z as the component forces *on unit mass*, and the previous equations become

$$m\frac{d^2x}{dt^2}=mX, \text{ \&c.};$$

from which m may of course be omitted.

[Some confusion is often introduced by the division of forces into "*accelerating*," and "*moving*," forces; and it is even stated occasionally that the former are of *one*, and the latter of *four* linear dimensions. The fact, however, is that an equation such as

$$\frac{d^2x}{dt^2}=X,$$

may be interpreted either as dynamical, or as merely kinematical. If kinematical, the meanings of the terms are obvious; if dynamical, the unit of mass must be understood as a factor on the left-hand side, and in that case X is the x-component per unit of mass, of the whole force exerted on the moving body.]

If there be no acceleration, we have of course equilibrium among the forces. Hence the equations of motion of a particle are changed into those of equilibrium by putting

$$\frac{d^2x}{dt^2}=0, \text{ \&c.}$$

65. We have, by means of the first two laws, arrived at a *definition* and a *measure* of force; and have also found

how to compound, and therefore also how to resolve, forces; and also how to investigate the conditions of equilibrium or motion of a single particle subjected to given forces. But more is required before we can completely understand the more complex cases of motion, especially those in which we have mutual actions between or amongst two or more bodies; such as, for instance, attractions or pressures or transference of energy in any form. This is perfectly supplied by

66. Law III. *To every action there is always an equal and contrary reaction: or, the mutual actions of any two bodies are always equal and oppositely directed in the same straight line.*

If one body presses or draws another, it is pressed or drawn by this other with an equal force in the opposite direction. If any one presses a stone with his finger, his finger is pressed with the same force in the opposite direction by the stone. A horse towing a boat on a canal is dragged backwards by a force equal to that which he impresses on the towing-rope forwards. By whatever amount, and in whatever direction, one body has its motion changed by impact upon another, this other body has its motion changed by the same amount in the opposite direction; for at each instant during the impact the force between them was equal and opposite on the two. When neither of the two bodies has any rotation, whether before or after impact, the changes of velocity which they experience are inversely as their masses. When one body attracts another from a distance, this other attracts it with an equal and opposite force.

67. We shall for the present take for granted, that the mutual action between two particles may in every case be imagined as composed of equal and opposite forces in the straight line joining them. From this it follows that the sum of the quantities of motion, parallel to any fixed direction, of the particles of any system influencing one another in any possible way, remains unchanged by their mutual action; also that the sum of the moments of momentum of all the particles round any line in a fixed direction in space, and passing through any point moving uniformly in a straight line in any direction, remains constant. From the first of these propositions we infer that the center of inertia of any

system of mutually influencing particles, if in motion, continues moving uniformly in a straight line, unless in so far as the direction or velocity of its motion is changed by forces acting mutually between the particles and some other matter not belonging to the system; also that the center of inertia of any system of particles moves just as all their matter, if concentrated in a point, would move under the influence of forces equal and parallel to the forces really acting on its different parts. From the second we infer that the axis of resultant rotation through the center of inertia of any system of particles, or through any point either at rest or moving uniformly in a straight line, remains unchanged in direction, and the sum of moments of momenta round it remains constant if the system experiences no force from without. [This principle is sometimes called *Conservation of Areas*, a very misleading designation.] These results will be deduced analytically in Chap. XII.

68. What precedes is founded upon Newton's own comments on the third law, and the actions and reactions contemplated are mere forces. In the scholium appended, he makes the following remarkable statement, introducing another specification of actions and reactions subject to his third law:—

Si æstimetur agentis actio ex ejus vi et velocitate conjunctim; et similiter resistentis reactio æstimetur conjunctim ex ejus partium singularum velocitatibus et viribus resistendi ab earum attritione, cohæsione, pondere, et acceleratione oriundis; erunt actio et reactio, in omni instrumentorum usu, sibi invicem semper æquales.

In a previous discussion Newton has shewn what is to be understood by the velocity of a force or resistance; *i.e.*, that it is the velocity of the point of application of the force *resolved in the direction of the force.* Bearing this in mind, we may read the above statement as follows:—

If the Action of an agent be measured by its amount and its velocity conjointly; and if, similarly, the Reaction of the resistance be measured by the velocities of its several parts and their several amounts conjointly, whether these arise from friction, cohesion, weight, or acceleration;—Action and Reaction, in all combinations of machines, will be equal and opposite.

69. Newton here points out that forces of resistance against acceleration are to be reckoned as reactions equal and opposite to the actions by which the acceleration is produced. Thus, if we consider any one material point of a system, its reaction against acceleration must be equal and opposite to the resultant of the forces which that point experiences, whether by the actions of other parts of the system upon it, or by the influence of matter not belonging to the system. In other words, it must be in equilibrium with these forces. Hence Newton's view amounts to this, that all the forces of the system, with the reactions against acceleration of the material points composing it, form groups of equilibrating systems for these points considered individually. Hence, by the principle of superposition of forces in equilibrium, all the forces acting on points of the system form, with the reactions against acceleration, an equilibrating set of forces on the whole system. This is the celebrated principle first explicitly stated, and very usefully applied by D'Alembert in 1742, and still known by his name.

Newton in the sentence just quoted lays, in an admirably distinct and compact manner, the foundations of the abstract theory of *Energy*, which recent experimental discovery has raised to the position of the grandest of known physical laws. He points out, however, only its application to mechanics. The *actio agentis*, as he defines it, which is evidently equivalent to the product of the effective component of the force, into the velocity of the point on which it acts, is simply, in modern English phraseology, the rate at which the agent works. The subject for measurement here is precisely the same as that for which Watt, a hundred years later, introduced the practical unit of a "*Horse-power*," or the rate at which an agent works when overcoming 33,000 times the weight of a pound through the space of a foot in a minute; that is, producing 550 foot-pounds of work per second. The unit, however, which is most generally convenient is that which Newton's definition implies, namely, the rate of doing work in which the unit of energy is produced in the unit of time.

70. Looking at Newton's words in this light, we see by § 46 that they may be logically converted into the following form:—

"Work done on any system of bodies (in Newton's statement, the parts of any machine) has its equivalent in work done against friction, molecular forces, or gravity, if there be no acceleration; but if there be acceleration, part of the work is expended in overcoming the resistance to acceleration, and the additional kinetic energy developed is equivalent to the work so spent."

When part of the work is done against molecular forces, as in bending a spring; or against gravity, as in raising a weight; the recoil of the spring, and the fall of the weight, are capable, at any future time, of reproducing the work originally expended (§ 57). But in Newton's day, and long afterwards, it was supposed that work was *absolutely lost* by friction.

71. If a system of bodies, given either at rest or in motion, be influenced by no forces from without, the sum of the kinetic energies of all its parts is augmented in any time by an amount equal to the whole work done in that time by the mutual forces, which we may imagine as acting between its points. When the lines in which these forces act remain all unchanged in length, the forces do no work, and the sum of the kinetic energies of the whole system remains constant. If, on the other hand, one of these lines varies in length during the motion, the mutual forces in it will do work, or will consume work, according as the distance varies with or against them.

72. Experiment has shewn that the mutual forces between the parts of any system of natural bodies always perform, or always consume, the same amount of work during any motion whatever, by which the system can pass from one particular configuration to another: so that each configuration corresponds to a definite amount of kinetic energy. [For the apparent violation of this by friction, impact, &c. see § 73*.] Hence no arrangement is possible, in which a gain of kinetic energy can be obtained when the system is restored to its initial configuration. In other words, "*the Perpetual Motion is impossible.*"

73.* The *potential energy* (§ 57) of such a system, in the

configuration which it has at any instant, is the amount of work that its mutual forces perform during the passage of the system from any one chosen configuration to the configuration at the time referred to. It is generally convenient so to fix the particular configuration, chosen for the zero of reckoning of potential energy, that the potential energy in every other configuration practically considered shall be positive.

To put this in an analytical form, we have merely to notice that by what has just been said, the value of

$$\Sigma \int \left(X \frac{dx}{ds} + Y \frac{dy}{ds} + Z \frac{dz}{ds} \right) ds$$

is independent of the paths pursued from the initial to the final positions, and therefore that

$$\Sigma \left(X dx + Y dy + Z dz \right)$$

is a complete differential. If, in accordance with what has just been said, this be called $-dV$, V is the potential energy, and

$$X_1 = -\frac{dV}{dx_1}, \; \ldots\ldots$$

Also, by the second law of motion, if m be the mass of a particle of the system whose co-ordinates are x, y, z, we have

$$m_1 \frac{d^2 x_1}{dt^2} = X_1, \;\; \&c. = \&c.$$

$$\text{and } \Sigma \left\{ m \left(\frac{dx}{dt} \frac{d^2 x}{dt^2} + \frac{dy}{dt} \frac{d^2 y}{dt^2} + \frac{dz}{dt} \frac{d^2 z}{dt^2} \right) \right\} dt = \Sigma (X dx + Y dy + Z dz)$$
$$= -dV.$$

The integral is

$$\frac{1}{2} \Sigma (m v^2) + V = C,$$

that is, *the sum of the kinetic and potential energies is constant*. This is called the *Conservation of Energy*.

In abstract dynamics, with which alone this treatise is concerned, there is loss of energy by friction, impact, &c. This we simply leave as loss, to be afterwards accounted for in physics.

73*. [The theory of energy cannot be completed until we are able to examine the physical influences which accompany loss of energy. We then see that in every case in which energy is lost by resistance, heat is generated; and we learn from Joule's investigations that the quantity of heat so generated is a perfectly definite equivalent for the energy lost. Also that in no natural action is there ever a development of energy which cannot be accounted for by the disappearance of an equal amount elsewhere by means of some known physical agency. Thus we conclude, that if any limited portion of the material universe could be perfectly isolated, so as to be prevented from either giving energy to, or taking energy from, matter external to it, the sum of its potential and kinetic energies would be the same at all times. But it is only when the inscrutably minute motions among small parts, possibly the ultimate molecules of matter, which constitute light, heat, and magnetism; and the intermolecular forces of chemical affinity; are taken into account, along with the palpable motions and measurable forces of which we become cognizant by direct observation, that we can recognise the universally conservative character of all natural dynamic action, and perceive the bearing of the principle of reversibility on the whole class of natural actions involving resistance, which seem to violate it. It is not consistent with the object of the present work to enter into details regarding transformations of energy. But it has been considered advisable to introduce the very brief sketch given above, not only in order that the student may be aware, from the beginning of his reading, what an intimate connection exists between Dynamics and the modern theories of Heat, Light, Electricity, &c.; but also that we may be enabled to use such terms as "*potential energy*," &c. instead of the unnatural "*Force-functions*," &c. which disfigure most of the modern analytical treatises on our subject.]

CHAPTER III.

RECTILINEAR MOTION.

74. THE simplest case of motion which we have to consider is that of a particle in a straight line. This may be due to a force acting at every instant in the direction of motion; or the particle may be supposed to be constrained to move in a straight line by its being enclosed in a straight tube of indefinitely small bore. As already mentioned, § 64, we shall in every case suppose the mass of the particle to be unity.

75. *A particle moves in a straight line, under the action of any forces, whose resultant is in that line; to determine the motion.*

Let P be the position of the particle at any time t, f the resultant acceleration acting always along OP, O being a fixed point in the line of motion.



Let $OP = x$, then the equation of motion is

$$\frac{d^2x}{dt^2} = f.$$

In this equation f may be given as a function of x, of $\frac{dx}{dt}$, or of t, or of any two or all three combined; but in any case the first and second integrals of the equation (if they can be obtained) will give $\frac{dx}{dt}$ and x in terms of t; that is, the position and velocity of the particle at any instant will be known.

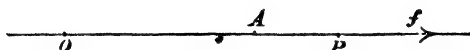
The only one of these cases which we will now consider is that in which f is given as a function of x; those in which f is a function of $\frac{dx}{dt}$, or of $\frac{dx}{dt}$ and x, being reserved for the

Chapter on Motion in a Resisting Medium: while those in which f involves t explicitly possess little interest, as they cannot be procured except by special adaptations; and can even then appear only in an incomplete statement of the circumstances of the particular arrangement.

The simplest supposition we can make is that f is constant.

76. *A particle, projected from a given point with a given velocity, is acted on by a constant force in the line of its motion; to determine the position and velocity of the particle at any time.*

Let A be the initial position of the particle, P its position at any time t, v its velocity at that time, and f the constant



acceleration of its velocity. Take any fixed point O in the line of motion as origin, and let $OA = a$, $OP = x$. The equation of motion is

$$\frac{d^2x}{dt^2} = f \qquad (1).$$

Integrating once, we have

$$\frac{dx}{dt} = v = ft + C,$$

C being a constant to be determined by the initial circumstances of the motion. Suppose the particle projected from A in the positive direction with velocity V, then when $t = 0$, $v = V$; hence $C = V$, and

$$\frac{dx}{dt} = v = V + ft \qquad (2).$$

Integrating again,

$$x = C' + Vt + f\frac{t^2}{2}.$$

But when $t=0$, $x=a$; hence $C'=a$, and

$$x = a + Vt + f\frac{t^2}{2} \ldots\ldots\ldots\ldots\ldots (3).$$

Equations (2) and (3) give the velocity and position of the particle in terms of t; and the velocity may be determined in terms of x by eliminating t between them: but the same result will be obtained more directly by multiplying (1) by $\frac{dx}{dt}$ and integrating. This gives the equation of energy

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{v^2}{2} = C'' + fx.$$

But when $x=a$, $v=V$; hence $C'' = \frac{V^2}{2} - fa$, and

$$\frac{v^2}{2} = \frac{V^2}{2} + f(x-a) \ldots\ldots\ldots\ldots\ldots (4).$$

77. The most important case of the motion of a particle under the action of a constant force in its line of motion is that in which the force is gravity. For the weights of bodies in the same latitude at small distances above the Earth's surface may be considered constant, and therefore if we denote the kinetic measure of the earth's attraction by g, and consider the particle to be projected vertically downwards; equations (2), (3), (4) of § 76 become

$$\left.\begin{aligned} v &= V + gt \\ x &= a + Vt + \tfrac{1}{2}gt^2 \\ \frac{v^2}{2} &= \frac{V^2}{2} + g(x-a) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (A),$$



x being measured as before from a fixed point O in the line of motion. As a particular instance suppose the particle to be dropped from rest at O. At that instant A coincides with O, and $a=0$, $V=0$.

$$\text{Hence } v = gt \text{ .... (1),}$$

$$x = \tfrac{1}{2}gt^2 \text{ .... (2),}$$

$$\frac{v^2}{2} = gx \text{ .... (3).}$$

The last of these equations may also be obtained from

$$g = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

by a single integration.

78. As another particular instance, suppose the particle to be projected vertically upwards. Here it must be remembered that if we measure x upwards from the point of projection, the force tends to diminish x and the equation of motion is

$$\frac{d^2x}{dt^2} = -g.$$

In other respects the solution is the same. Taking, therefore, $a = 0$ in equations (A) and changing the sign of g, we obtain

$$v = V - gt \text{ .... (4),}$$

$$x = Vt - \frac{gt^2}{2} \text{ .... (5),}$$

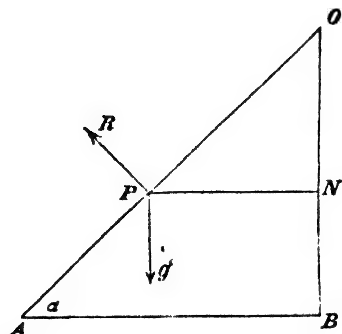
$$\frac{v^2}{2} = \frac{V^2}{2} - gx \text{ .... (6).}$$

From equation (4) we see that the velocity continually diminishes, and becomes zero when $t = \frac{V}{g}$; and from (6) that the height corresponding to $v = 0$, or the greatest height to which the particle will ascend, is $\frac{V^2}{2g}$. After this the velocity becomes negative, or the particle begins to descend, and (5) shews that it will return to the point of projection when $t = \frac{2V}{g}$, as x then becomes 0; and the velocity with which

it returns to that point is, by (6), equal to the velocity of projection.

79. *A particle descends a smooth inclined plane under the action of gravity, the motion taking place in a vertical plane perpendicular to the intersection of the inclined with any horizontal plane; to determine the motion.*

Let P be the position of the particle at any time t on the inclined plane OA, $OP = x$ its distance from a fixed point O



in the line of motion, and let α be the inclination of OA to the horizontal line AB. The only impressed force on the particle is its weight g which acts vertically downwards, and this may be resolved into two, $g \sin \alpha$ along, and $g \cos \alpha$ perpendicular to, OA. Besides these there is the unknown force R, or the reaction of the plane, which is perpendicular to OA: but neither this nor the component $g \cos \alpha$ can affect the motion along the plane. The equation of motion is therefore

$$\frac{d^2x}{dt^2} = g \sin \alpha,$$

the solution of which, as $g \sin \alpha$ is constant, is included in that of the proposition of § 77, and all the results for particles moving vertically under the action of gravity will be made to apply to it by writing $g \sin \alpha$ for g. Thus, if the particle

start from rest at O, we get from equations (1), (2), (3) of § 77 by this means,

$$v = g \sin \alpha \,.\, t \text{ .... (1)},$$

$$x = \tfrac{1}{2} g \sin \alpha \,.\, t^2 \text{ .... (2)},$$

$$\frac{v^2}{2} = g \sin \alpha \,.\, x \text{ .... (3)}.$$

80. Equation (3) proves an important property with regard to the velocity acquired at any point of the descent. For, draw PN parallel to AB, and let it meet the vertical line through O in N, then if v be the velocity at P, we have

$$\frac{v^2}{2} = g \sin \alpha \,.\, OP$$
$$= g \,.\, ON.$$

Comparing this with equation (3) of § 77, we see that the velocity at P is the same as that which a particle would acquire by falling freely from rest through the vertical distance ON; in other words the velocity at any point, of a particle sliding down a smooth inclined plane, is that due to the vertical height through which it has descended; a particular case of the conservation of energy.

81. Again from (2) we derive immediately the following curious and useful result.

The times of descent down all chords drawn through the highest or lowest point of a vertical circle are equal.

Let AB be the vertical diameter of the circle, AC any chord through A; join BC; then if T be the time of descent down AC, we have by equation (2) of § 79,

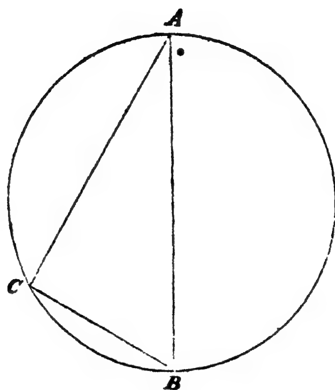
$$AC = \tfrac{1}{2} g T^2 \cos BAC.$$

But $AC = AB \cos BAC$; whence

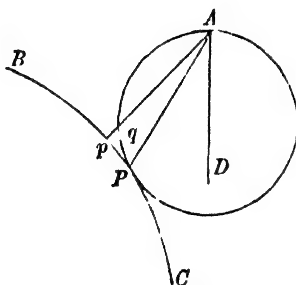
$$AB = \tfrac{1}{2} g T^2,$$

$$\text{or } T = \sqrt{\frac{2AB}{g}},$$

which, being independent of the position of the chord, gives the same time of descent for all.



It may similarly be shewn that the time of descent down all chords through B is the same.



To find the straight line of swiftest descent to a given curve from any point in the same vertical plane, all that is required is to draw a circle having the given point as the upper extremity of its vertical diameter, and the smallest which can meet the curve. Hence if BC be the curve, A the point, draw AD vertical; and, with center in AD, describe a circle passing through A and touching BC. Let P be the point of contact, then AP is the required line. For, if we take any

other point, p, in BC, Ap cuts the circle in some point q, and time down $Ap >$ time down Aq, i. e. $>$ time down AP.

If the given curve be not plane, a sphere must be described passing through A, with center in AD, so as to touch the curve; and the proof is precisely as before.

82. In § 79 we have supposed the inclined plane to be smooth, but the motion will still be uniformly accelerated when the plane is rough. For since there is no motion perpendicular to OA (see fig. § 79), we must have

$$R = g \cos \alpha.$$

If μ then be the coefficient of kinetic friction, which is known by experiment to be independent of the velocity of the particle, the retarding force of friction will be μR or $\mu g \cos \alpha$, and the equation of motion will become

$$\frac{d^2x}{dt^2} = g \sin \alpha - \mu g \cos \alpha,$$

the second member still being constant, and the solution therefore similar to those we have already considered.

83. *When a particle moves under the action of a force in its line of motion, the force varying directly as the distance of the particle from a fixed point in that line, to determine the motion.*

Let O be the fixed point, P the position of the particle at any time t, v its velocity at that time, and let $OP = x$. Then



if μ be the acceleration of a particle at a unit of distance from O, which is supposed known, the acceleration at P will be μx, and if it be directed towards O will tend to diminish x. Therefore the equation of motion is

$$\frac{d^2x}{dt^2} = -\mu x,$$

$$\text{or } \frac{d^2x}{dt^2} + \mu x = 0 \text{ .... (1).}$$

Multiplying this equation by $\frac{dx}{dt}$, and integrating, we obtain

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{\mu}{2}(A^2 - x^2) \text{ .... (2),}$$

the equation of energy. This may be written

$$\frac{dt}{dx} = -\frac{1}{\sqrt{\mu}} \frac{1}{\sqrt{(A^2-x^2)}};$$

the negative sign being employed if we suppose the motion to be towards O, and A being the constant introduced in the integration. Integrating again

$$\sqrt{\mu}t + B = \cos^{-1}\frac{x}{A};$$

$$\text{or } x = A \cos\{\sqrt{\mu}t + B\} \text{ .... (3),}$$

the complete integral of (1); involving two arbitrary constants A and B, the values of which are to be determined from the initial distance, and the velocity of projection. Thus from (3),

$$\frac{dx}{dt} = v = -\sqrt{\mu}A \sin\{\sqrt{\mu}t + B\} \text{ .... (4).}$$

84. Suppose the particle to be projected from A in the positive direction with the velocity V, and let $OA = a$; then when $t = 0$, we have $x = a$, $v = V$; and therefore from (3) and (4)

$$a = A \cos B,$$

$$V = -\sqrt{\mu}A \sin B,$$

which determine A and B, and then (3) and (4) give the position and velocity of the particle at any instant. The velocity in terms of x is obtained directly from (2), for when $x = a$, we have $v = V$; whence $V^2 = \mu(A^2 - a^2)$, and

$$v^2 = V^2 + \mu(a^2 - x^2).$$

85. Equations (3) and (4) give periodical values of x and v, such that all the circumstances of motion are the same at the time $t+\frac{2\pi}{\sqrt{\mu}}$ as at the time t. They also shew that the velocity becomes zero when $\sqrt{\mu}t+B=0$, and that the corresponding value of x is the greatest possible. Hence the particle will move in the positive direction to a distance A from O, and then begin to return. Also, since when $\sqrt{\mu}t+B=\pm\pi$, we have $v=0$ again, and $x=-A$, it will pass through O, move to an equal distance on the other side, and so on: the time of a complete oscillation, that is, the time from its leaving any point until it passes through it again in the same direction, being $\frac{2\pi}{\sqrt{\mu}}$. This result is remarkable, as it shews that the time of oscillation is independent of the velocity and distance of projection, and depends solely on the intensity of the force.

The above proposition includes the motion of a particle within a homogeneous sphere of ordinary matter, in a straight bore to the center. For the attraction of such a sphere on a particle within it is proportional to the distance from the center, and the equation of motion is therefore the same as that which we have just considered.

Suppose O itself to be in motion in the line OA, and let ξ denote its position at time t. The equation of motion is

$$\frac{d^2x}{dt^2}=-\mu(x-\xi),$$

and is integrable when ξ is given in terms of t. This may be at once changed into the equation of relative motion

$$\frac{d^2(x-\xi)}{dt^2}=-\mu(x-\xi)-\frac{d^2\xi}{dt^2},$$

which is the same as when the point O is at rest if $\frac{d^2\xi}{dt^2}=0$, i.e. if the velocity of O be constant. If O move with constant acceleration, α, the oscillatory motion will be the same as before, but the mean position will be $\frac{\alpha}{\mu}$ behind O.

86. *If the force in § 83 be supposed repulsive or directed always from the center instead of towards it, the equation of motion becomes*

$$\frac{d^2x}{dt^2} = \mu x,$$

the integral of which is known to be

$$x = A\epsilon^{\sqrt{\mu}t} + B\epsilon^{-\sqrt{\mu}t};$$

and the motion is not oscillatory. If, when $t=0$, $x=B$, $v=-B\sqrt{\mu}$, the particle constantly approaches the centre but never reaches it.

87. It is to be remarked that we cannot always apply the same equation of motion to the negative and positive sides of the origin as we have done in the case of § 83. Our being able to do so arises from the fact that the expression, μx, for the force changes sign with x; for by looking at the figure it will be seen that when x is negative the force tends to increase x algebraically, and the equation ought properly to be written

$$\frac{d^2x}{dt^2} = +\mu(-x).$$

In general, when the force is proportional to the n^{th} power of the distance, the equations of motion for the positive and negative sides of the origin are respectively

$$\frac{d^2x}{dt^2} = \mu x^n,$$

$$\text{and } \frac{d^2x}{dt^2} = -\mu(-x)^n.$$

The only cases, therefore, in which the same equation of motion will apply to both sides of the origin, occur when n is of the form $\frac{2m+1}{2m'+1}$, where m, m' are any whole numbers including zero, since it is only in these cases that we have

$$-(-x)^n = x^n.$$

88. In all other cases the investigation of the motion will generally consist of two parts, one for each side of the origin;

and in one case even when n is of the form $\frac{2m+1}{2m'+1}$ *it is* necessary to consider these parts separately, because the form *of the integral is not sufficiently general to include both.* This is when $m=0$ and $m'=-1$, for in that case the equation of motion becomes

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x}.$$

Multiplying this by $2\frac{dx}{dt}$ and integrating we have

$$\left(\frac{dx}{dt}\right)^2=C-2\mu\log x,$$

which becomes impossible when x is negative. But it is evident that we may then write the integral

$$\left(\frac{dx}{dt}\right)^2=C-2\mu\log(-x),$$

which is, of course, the proper form for the negative side of the origin. These equations cannot generally be integrated farther, but we will shew towards the end of the Chapter how the time of reaching the origin may be determined.

89. *A particle, constrained to move in a straight line, is acted on by a force always directed to a point outside the line, and varying directly as the distance of the particle from that point, to determine the motion.*

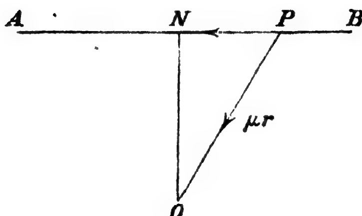
The constraint here contemplated may be conceived by considering the particle either as an indefinitely small ring sliding on a smooth rod, or as a material particle sliding in a smooth straight tube of indefinitely small bore.

Let AB be the straight line, P the position of the particle at any time, O the point to which the force on P is always directed. Draw ON perpendicular to AB, and let $NP=x$; then if $OP=r$, and if μ as formerly be the acceleration at a unit of distance, the acceleration of P along PO is μr. This may be resolved into two, one along and the other perpendicular to AB, of which the latter has no effect on the motion

of the particle. The equation of motion is, therefore, since the acceleration is $\mu r \cos OPN$ or μPN,

$$\frac{d^2x}{dt^2} = -\mu x,$$

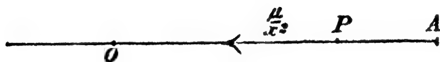
the same as in § 83. The motion of the particle will therefore be oscillatory about N, the time of a complete oscillation



being $\frac{2\pi}{\sqrt{\mu}}$, and all the circumstances of motion the same as for a free particle moving in AB under the action of an equal center of force placed at N.

90. *A particle moves in a straight line under the action of a force always directed to a point in that line and varying inversely as the square of the distance from that point; to determine the motion.*

Let O be the fixed point, P the position of the particle at



the time t, $OP = x$; the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2};$$

μ being, as before, the acceleration at unit distance from O.

Multiplying by $\frac{dx}{dt}$ and integrating, we get

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{v^2}{2} = C + \frac{\mu}{x},$$

the equation of energy.

Suppose the particle to start from rest at a point A distant a from O, then when $x = a$, $v = 0$;

hence, $$C = -\frac{\mu}{a}, \text{ and}$$

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{v^2}{2} = \mu\left(\frac{1}{x} - \frac{1}{a}\right) \text{ .... } (1),$$

which gives the velocity of the particle at any distance x from the origin. Again from (1)

$$\frac{dx}{dt} = -\sqrt{2\mu}\sqrt{\frac{a-x}{ax}},$$

the negative sign being taken, since in the motion towards O, x diminishes as t increases. This gives

$$\frac{dt}{dx} = -\sqrt{\frac{a}{2\mu}} \cdot \frac{x}{\sqrt{(ax - x^2)}}$$

$$= \sqrt{\frac{a}{2\mu}}\left\{\frac{1}{2}\frac{a-2x}{\sqrt{(ax-x^2)}} - \frac{a}{2}\frac{1}{\sqrt{(ax-x^2)}}\right\}.$$

Integrating, we have

$$t = \sqrt{\frac{a}{2\mu}} \cdot \left\{\sqrt{(ax - x^2)} - \frac{a}{2}\operatorname{vers}^{-1}\frac{2x}{a} + C'\right\}.$$

Now, when $t = 0$, $x = a$, and therefore $C' = \dfrac{\pi a}{2}$.

Hence $$\sqrt{\frac{2\mu}{a}}\, t = \sqrt{(ax - x^2)} - \frac{a}{2}\operatorname{vers}^{-1}\frac{2x}{a} + \frac{\pi a}{2},$$

which is the relation between x and t.

91. Putting $x = 0$, we find that the time of arriving at O is $\dfrac{\pi}{2}\sqrt{\dfrac{a^3}{2\mu}}$, and (1) shews that the velocity at O is in-

finite. On this account we are precluded from applying our formulæ to determine the motion after arriving at O; but it is to be observed that, although at any point very near to O there is a very great force tending towards O, at the point O itself there is no force at all: and therefore the particle, approaching the center of force with an indefinitely great velocity, must pass through it. Also, everything being the same at equal distances on either side of the center, we see that the motion must be checked as rapidly as it was generated, and therefore the particle will proceed to a distance on the other side of S equal to that from which it started. The motion will then continue oscillatory.

92. The above case of motion includes that of a body falling from a great height above the Earth's surface. For a sphere attracts an external particle with a force varying inversely as the square of the distance of the particle from its center, and therefore if x be the distance of a body from the Earth's center, R the Earth's radius, and g the kinetic measure of gravity on unit of mass at the Earth's surface, the equation of motion will be

$$\frac{d^2x}{dt^2} = -g\frac{R^2}{x^2},$$

the same equation as before, if we write μ for gR^2. The results just obtained will therefore apply to this case. Thus if we wish to find the velocity which a body would acquire in falling to the Earth's surface from a height h above it, we have from (1), putting $\mu = gR^2$,

$$\frac{v^2}{2} = gR^2\left(\frac{1}{x} - \frac{1}{R+h}\right);$$

and therefore if V be the velocity when $x = R$, i.e. the required velocity,

$$\frac{1}{2}V^2 = gR\frac{h}{R+h}.$$

If h be small compared with R, this may be written

$$\frac{1}{2}V^2 = gh\left(1 - \frac{h}{R} + \&c.\right)$$

from which we see the amount of error introduced by the ordinary formula, § 77,

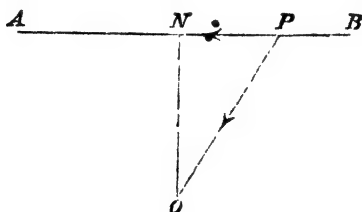
$$\frac{1}{2}V^2=gh.$$

If the fall be from an infinite distance, $h=\infty$, and we have

$$\frac{1}{2}V^2=gR.$$

93. *A particle is constrained to move in a straight line, and is acted on by a force, always directed to a point outside that line, and varying inversely as the square of the distance from that point; to determine the motion.*

Let AB be the straight line, P the position of the particle at any time, O the point to which the force is always directed,



μ the acceleration at unit distance. Draw ON perpendicular to AB and let $ON=b$, $NP=x$; then the acceleration of P along PO is $\frac{\mu}{PO^2}$, and, as in § 89, the only part of this which produces motion is the resolved part along PN. Therefore the equation of motion is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{OP^2}\cos OPN$$

$$=-\frac{\mu x}{(x^2+b^2)^{\frac{3}{2}}} \text{ ....(1)}.$$

Multiplying by $2\frac{dx}{dt}$ and integrating, we have

$$\left(\frac{dx}{dt}\right)^2 = v^2 = C + \frac{2\mu}{(x^2+b^2)^{\frac{1}{2}}},$$

where C is to be determined in the usual manner.

94. This equation cannot generally be integrated farther, but in this and every similar case the integration can be performed if we suppose x always very small. Suppose the particle to have been at rest at N, and to have been slightly displaced from this position of equilibrium, the displacement being so small that throughout the motion $\frac{x^2}{b^2}$ may be neglected in comparison with $\frac{x}{b}$. We have from (1),

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{b^3}\left(1+\frac{x^2}{b^2}\right)^{-\frac{3}{2}}$$

$$= -\frac{\mu x}{b^3}\left(1 - \frac{3}{2}\frac{x^2}{b^2} + \&c.\right)$$

$$= -\frac{\mu x}{b^3} \text{ nearly;}$$

$$\text{or} \quad \frac{d^2x}{dt^2} + \frac{\mu x}{b^3} = 0,$$

the same form of equation of motion as that of § 83. The motion will therefore be oscillatory, the time of each small oscillation being $2\pi\sqrt{\frac{b^3}{\mu}}$.

95. *A particle moves in a straight line under the action of a force varying inversely as the* n^th^ *power of the distance of the particle from a fixed point in that line; to determine the motion.*

Measuring x as before, the equation of motion will be

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^n}.$$

Multiplying by $2\frac{dx}{dt}$ and integrating, we have

$$\left(\frac{dx}{dt}\right)^2 = v^2 = C + \frac{2}{n-1}\frac{\mu}{x^{n-1}}.$$

Suppose the particle to start from rest at a distance a from the fixed point; then when $x = a$, $v = 0$; therefore

$$C = -\frac{2}{n-1}\frac{\mu}{a^{n-1}};$$

$$\text{and} \quad \left(\frac{dx}{dt}\right)^2 = v^2 = \frac{2\mu}{n-1}\left(\frac{1}{x^{n-1}} - \frac{1}{a^{n-1}}\right) \quad \text{.............. (1).}$$

96. This equation cannot generally be integrated farther, but if we suppose the particle to have started from a point at an infinite distance, we have $a = \infty$, and

$$v^2 = \frac{2\mu}{n-1}\frac{1}{x^{n-1}},$$

where v is the velocity from infinity, at the distance x.

We have therefore in this particular case

$$\frac{dx}{dt} = \left(\frac{2\mu}{n-1}\right)^{\frac{1}{2}} \frac{1}{x^{\frac{n-1}{2}}},$$

$$\text{or} \quad \frac{dt}{dx} = \left(\frac{n-1}{2\mu}\right)^{\frac{1}{2}} x^{\frac{n-1}{2}}.$$

Integrating this between the limits $x = \alpha$, $x = \beta$, we have for the time of moving from $x = \alpha$ to $x = \beta$,

$$T = \frac{2}{n+1}\left(\frac{n-1}{2\mu}\right)^{\frac{1}{2}} \left(\alpha^{\frac{n+1}{2}} - \beta^{\frac{n+1}{2}}\right).$$

97. If we expand (1) into a series, we obtain

$$\frac{dt}{dx} = -\left(\frac{n-1}{2\mu}\right)^{\frac{1}{2}} x^{\frac{n-1}{2}} \left(1 - \frac{x^{n-1}}{a^{n-1}}\right)^{-\frac{1}{2}}$$

$$= -\left(\frac{n-1}{2\mu}\right)^{\frac{1}{2}} x^{\frac{n-1}{2}} \left(1 + \frac{1}{2}\frac{x^{n-1}}{a^{n-1}} + \frac{1.3}{2.4}\frac{x^{2n-2}}{a^{2n-2}} + \&c.\right).$$

Integrating between the limits $x = a$, $x = 0$, we get for the time of falling to the center from a distance a the expression

$$\left(\frac{n-1}{2\mu}\right)^{\frac{1}{2}} 2a^{\frac{n+1}{2}} \left(\frac{1}{n+1} + \frac{1}{2}\frac{1}{3n-1} + \&c.\right),$$

which therefore for different distances varies as $a^{\frac{n+1}{2}}$.

Or, better, thus. Put $\frac{x}{a} = z$, and we have, for the time of fall to the center from rest at distance a, the expression

$$\left(\frac{n-1}{2\mu}\right)^{\frac{1}{2}} a^{\frac{n+1}{2}} \int_0^1 \frac{z^{\frac{n-1}{2}} dz}{(1-z^{n-1})^{\frac{1}{2}}} = \left\{\frac{1}{2\mu(n-1)}\right\}^{\frac{1}{2}} a^{\frac{n+1}{2}} \mathsf{F}\left\{\frac{n+1}{2(n-1)}, \frac{1}{2}\right\},$$

where F is "Euler's first integral."

98. The above solution fails when $n = 1$, but the time of falling to the center may be found as follows. The equation for this case, as given in § 88, is

$$\left(\frac{dx}{dt}\right)^2 = C - 2\mu \log x$$

$$= 2\mu \log \frac{a}{x},$$

since when $x = a$, $\frac{dx}{dt} = 0$. Hence,

$$\sqrt{2\mu}\frac{dt}{dx} = -\frac{1}{\sqrt{\log \frac{a}{x}}},$$

the negative sign being taken since x diminishes as t increases. Put T for the required time, then

$$\sqrt{2\mu}\, T = -\int_a^0 \frac{dx}{\sqrt{\log \frac{a}{x}}}.$$

To transform the integral, put $\sqrt{\log \frac{a}{x}} = y$. Then we have

$$x = a\epsilon^{-y^2}, \text{ and } \frac{dx}{dy} = -2a\epsilon^{-y^2}y,$$

and the limits of y are 0 and ∞. Hence

$$\sqrt{2\mu}\,.\,T = 2a\int_0^\infty \epsilon^{-y^2}\,dy,$$

which (*Gregory's Examples*, p. 466)

$$= 2a\,.\,\tfrac{1}{2}\sqrt{\pi}.$$

Hence $$T = a\sqrt{\frac{\pi}{2\mu}},$$

and is therefore directly as the space traversed.

99. *A particle is constrained to move in a straight line, and is acted on by a force directed to a point not in that line, and expressed by a function $\phi(r)$ of the distance; to determine the time of a small oscillation.*

Employing the same notation as in § 93, the acceleration along PO being $\phi(r)$, its component along PN is $\phi(r)\frac{x}{r}$; therefore the equation of motion is

$$\frac{d^2x}{dt^2} = -\phi(r)\frac{x}{r}.$$

But $$r = \sqrt{(b^2 + x^2)} = b\sqrt{\left(1 + \frac{x^2}{b^2}\right)}$$

$$= b \text{ approximately}.$$

Hence $$\frac{d^2x}{dt^2} + \frac{\phi(b)}{b} x = 0,$$

and therefore by § 85, the time of a small oscillation is

$$2\pi \sqrt{\frac{b}{\phi(b)}}.$$

EXAMPLES.

(1) A body is projected vertically upwards with a velocity which will carry it to a height $2g$ feet; shew that after three seconds it will be descending with a velocity g.

(2) Find the position of a point on the circumference of a vertical circle, in order that the time of rectilinear descent from it to the center may be the same as the time of descent to the lowest point.

(3) The straight line down which a particle will slide in the shortest time from a given point to a given circle in the same vertical plane, is the line joining the point to the upper or lower extremity of the vertical diameter, according as the point is within or without the circle.

(4) Find the locus of all points from which the time of rectilinear descent to each of two given points is the same. Shew also that in the particular case in which the given points are in the same vertical, the locus is formed by the revolution of a rectangular hyperbola.

(5) Find the line of quickest descent from the focus to a parabola whose axis is vertical and vertex upwards, and shew that its length is equal to that of the latus rectum.

(6) Find the straight line of quickest descent from the focus of a parabola to the curve when the axis is horizontal.

(7) The locus of all points in the same vertical plane for which the least time of sliding down an inclined plane to a circle is constant is another circle.

(8) Two bodies fall in the same time from two given points in space in the same vertical down two straight lines drawn to any point of a surface, shew that the surface is an equilateral hyperboloid of revolution, having the given points as vertices.

(9) Find the form of a curve in a vertical plane, such that if heavy particles be simultaneously let fall from each point of it so as to slide freely along the normal at that point, they may all reach a given horizontal straight line at the same instant.

(10) A semicycloid is placed with its axis vertical and vertex downwards, and from different points in it a number of particles are let fall at the same instant, each moving down the tangent at the point from which it sets out; prove that they will reach the involute (which passes through the vertex) all at the same instant.

(11) A particle moves in a straight line under the action of a force varying inversely as the $\left(\frac{3}{2}\right)^{\text{th}}$ power of the distance, shew that the velocity acquired by falling from an infinite distance to a distance a from the center is equal to the velocity which would be acquired in moving from rest at a distance a to a distance $\frac{a}{4}$.

(12) A particle moves in a straight line from a distance a towards a center of force, the force varying inversely as the cube of the distance; shew that the whole time of descent

$$=\frac{a^2}{\sqrt{\mu}}.$$

(13) A particle is placed at a given point between two centers of force of equal intensity attracting directly as the distance; to determine the motion and the time of an oscillation.

Let $2a$ be the distance between the centers, x the distance of the particle at any time from the middle point between them, then the equation of motion is

$$\frac{d^2x}{dt^2} = -\mu(a+x) + \mu(a-x)$$
$$= -2\mu x.$$

Hence, the time of an oscillation $= \dfrac{\pi}{\sqrt{(2\mu)}}$.

(14) If a particle begin to move directly towards a fixed center which repels with a force $= \mu$ (distance), and with an initial velocity $= \mu^{\frac{1}{2}}$ (initial distance), prove that it will continually approach the fixed center, but never attain to it.

(15) A particle acted upon by two centers of force, each attracting with an intensity varying inversely as the square of the distance, is projected from a given point between them, to find the velocity of projection that the particle may just arrive at the neutral point of attraction and remain at rest there.

If μ, μ' be the absolute forces of the centers; a_1, a_2 the distances of the point of projection from them; and V the initial velocity; we have

$$V^2 = \frac{(\sqrt{\mu a_2} - \sqrt{\mu' a_1})^2}{a_1 a_2 (a_1 + a_2)}.$$

(16) Supposing the Earth a homogeneous spheroid of equilibrium, the time of descent of a body let fall from any point P on the surface down a hole bored to the center C, varies as CP, and the velocity at the center is constant.

(17) A material particle placed at a center of attraction varying as the distance, is urged from rest by a constant force which acts for one-sixth of the time of a complete oscillation about the center, ceases for the same period, and then acts as before, shew that the particle will then be retained at rest, and that the spaces moved through in the two periods are equal.

(18) A body moves from rest at a distance a towards a center of force, the force varying inversely as the distance: shew that the time of describing the space between βa and $\beta^n a$ will be a maximum if $\beta = \dfrac{1}{n^{\frac{1}{2(n-1)}}}$.

(19) If the time of a body's descent in a straight line towards a given center of force vary inversely as the square of the distance fallen through, determine the law of the force.

(20) Assuming the velocity of a body falling to a center of force to be as $\sqrt{\dfrac{a-x}{x}}$, where a is the initial and x the variable distance from the center, find the law of the force.

(21) Find the time of falling to the center when the force $\propto$ (dist.)$^{-\frac{5}{3}}$.

(22) Shew that the time of descent, to a center of force $\propto$ (dist.)$^{-2}$, through the first half of the initial distance, is to that through the last half as $\pi + 2 : \pi - 2$.

(23) A particle descends to a center of force $\propto$ (dist.)^{n}. Find n so that the velocity acquired from infinity to distance a, shall be equal to that acquired from distance a to distance $\frac{1}{4}a$, from the center.

(24) A particle is placed at the extremity of the axis of a thin attracting cylinder of infinite length and of radius a, shew that its velocity after describing a space x is proportional to

$$\sqrt{\log \frac{x + \sqrt{(x^2 + a^2)}}{a}}.$$

(25) A particle falls to an infinite homogeneous solid bounded by a plane face, find the time of descent.

(26) Every point of a fine uniform ring repels with a force $\propto$ (dist.)$^{-2}$, find the time of a small oscillation in its plane, about the center.

(27) Shew that a body cannot move so that the velocity shall vary as the space from the beginning of the motion. And if the velocity vary as the cube root of that space, determine the force, and the time of describing a given space.

(28) Shew that the time of quickest descent down a focal chord of a parabola whose axis is vertical is

$$\sqrt{\frac{3^{\frac{3}{2}} l}{g}},$$

where l is the latus rectum.

(29) An ellipse is suspended with its major axis vertical, find the diameter down which a particle will fall in the least time, and the limiting value of the excentricity that this may not be the axis major itself.

(30) Particles slide down chords from a point O to a curved surface, under the action of a plane whose attraction is as the distance, and they reach the surface in the same time; shew that the surface is generated by the revolution (about a line whose length is a through O perpendicular to the plane) of the curve whose polar equation about O is

$$\rho \cos \theta = a \{1 - \cos (k \cos \theta)\}.$$

(31) If the particles commence their motion at the surface, and reach O after a given time, the equation to the generating curve is

$$\rho \cos \theta = a \{\sec (k \cos \theta) - 1\}.$$

(32) Prove that the times of falling through a given space AC towards a center of force S, under the action of two forces, one of which varies as the distance, and the other is constant and equal to the original value of the first, are as the arc (whose versed sine is AC) to the chord, in a circle whose radius is AS.

(33) The earth being supposed a thin uniform spherical shell, in the surface of which a circular aperture of given radius is made, if a particle be dropped from the center of the aperture, determine its velocity at any point of the descent.

(34) If a particle fall down a radius of a circle under the action of a force $\propto (D)^3$ in the center, and ascend the opposite radius under the action of the same force supposed repulsive, shew that it will acquire a velocity which is a geometric mean between radius, and the force at the circumference.

(35) If a particle fall to a center of force $\propto (D)$; determine the constant force which would produce the effect in the same time, and compare the final velocities.

(36) Find the equation to the curve down each of whose tangents a particle will slide to the horizontal axis in a given time.

(37) A sphere is composed of an infinite number of free particles, equally distributed, which gravitate to each other without interfering; supposing the particles to have no initial velocity, prove that the mean density about a given particle will vary inversely as the cube of its distance from the center.

CHAPTER IV.

PARABOLIC MOTION.

100. In this chapter we intend to treat principally of the motion of a free particle which is subject to the action of forces whose resultant is parallel to a given fixed line.

The simplest case of course will be when that resultant is constant. The problem then becomes the determination of the motion of a projectile in vacuo, since the attraction of the earth may be considered within moderate limits as constant and parallel to a fixed line. This we will now consider.

101. *A free particle moves under the action of a vertical force whose magnitude is constant; to determine the form of the path, and the circumstances of its description.*

Taking the axis of x horizontal and in the vertical plane and sense of projection, and that of y vertically upwards, it is evident that the particle will continue to move in the plane of xy, as it is projected in it, and is subject to no force which would tend to withdraw it from that plane.

The equations of motion then are

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = -g,$$

if g be the kinetic measure of the force.

Suppose that the point from which the particle is projected is taken as origin, that the velocity of projection is V, and that the direction of projection makes an angle α with the axis of x.

The first and second integrals of the above equations will then be

$$\frac{dx}{dt} = V\cos\alpha, \quad \frac{dy}{dt} = V\sin\alpha - gt \text{ .... } (1).$$

$$x = V\cos\alpha \, . \, t, \quad y = V\sin\alpha \, . \, t - \tfrac{1}{2}gt^2 \text{ .... } (2).$$

These equations give the co-ordinates of the particle and its velocity parallel to either axis for any assumed value of the time.

Eliminating t between equations (2) we obtain the equation to the trajectory, viz.

$$y = x \tan \alpha - \frac{g}{2V^2 \cos^2 \alpha} x^2 \text{ .... (3)},$$

which shews that the particle will move in a parabola whose axis is vertical, and vertex upwards.

102. Equation (3) may be written

$$x^2 - \frac{2V^2 \sin \alpha \cos \alpha}{g} x = -\frac{2V^2 \cos^2 \alpha}{g} y,$$

$$\text{or } \left(x - \frac{V^2 \sin \alpha \cos \alpha}{g}\right)^2 = -\frac{2V^2 \cos^2 \alpha}{g}\left(y - \frac{V^2 \sin^2 \alpha}{2g}\right).$$

By comparing this with the equation to a parabola, we find for the co-ordinates x_0, y_0 of the vertex

$$x_0 = \frac{V^2 \sin \alpha \cos \alpha}{g}, \quad y_0 = \frac{V^2 \sin^2 \alpha}{2g}.$$

Hence we obtain the equation to the directrix

$$y = y_0 + \tfrac{1}{4}\,(\text{parameter}) = \frac{V^2 \sin^2 \alpha}{2g} + \frac{V^2 \cos^2 \alpha}{2g} = \frac{V^2}{2g}.$$

Now if v be the velocity of the particle at any point of its path,

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2, \text{ or by (1),}$$

$$= (V^2 \cos^2 \alpha) + (V^2 \sin^2 \alpha - 2Vg \sin \alpha \,.\, t + g^2 t^2)$$

$$= V^2 - 2g\,(V \sin \alpha \,.\, t - \tfrac{1}{2} g t^2)$$

$$= V^2 - 2gy, \text{ by (2).}$$

To acquire this velocity in falling, from rest, the par-

ticle must have fallen, § 77, through a height $\frac{v^2}{2g}$, or $\frac{V^2}{2g} - y$, i.e. through the distance from the directrix.

103. *To find the time of flight along a horizontal plane.*

Put $y = 0$ in equation (3). The corresponding values of x are 0 and $\frac{2V^2}{g} \sin \alpha \cos \alpha$. But the horizontal velocity is $V \cos \alpha$. Hence the time of flight is $\frac{2V \sin \alpha}{g}$; and, ceteris paribus, varies as the sine of the inclination to the horizon of the direction of projection.

104. *To find the time of flight along an inclined plane passing through the point of projection.*

Let its intersection with the plane of projection make an angle β with the horizon; it is evident that we have only to eliminate y between (3) and $y = x \tan \beta$.

This gives for the abscissa of the point where the projectile meets the plane,

$$x_1 = \frac{2V^2}{g} (\sin \alpha \cos \alpha - \tan \beta \cos^2 \alpha)$$

$$= \frac{2V^2 \cos \alpha \sin (\alpha - \beta)}{g \cos \beta}.$$

Hence time of flight

$$= \frac{x_1}{V \cos \alpha} = \frac{2V}{g} \frac{\sin (\alpha - \beta)}{\cos \beta}.$$

105. *To find the direction of projection which gives the greatest range on a given plane.*

The range on the horizontal plane is $\frac{V^2}{g} \sin 2\alpha$. For a given value of V this will be greatest when

$$2\alpha = \frac{\pi}{2}, \text{ or } \alpha = \frac{\pi}{4}.$$

That on the inclined plane is $\frac{x_1}{\cos\beta}$, or

$$\frac{2V^2}{g\cos^2\beta}\cos\alpha\sin(\alpha-\beta).$$

That this may be a maximum for a given value of V we must equate to zero its differential coefficient with respect to α, which gives the equation

$$\cos\alpha\cos(\alpha-\beta)-\sin\alpha\sin(\alpha-\beta)=0,$$

$$\text{or } \cos(2\alpha-\beta)=0;$$

$$\text{whence } \alpha=\frac{1}{2}\left(\frac{\pi}{2}+\beta\right).$$

Hence the direction of projection required for the greatest range makes with the vertical an angle

$$\frac{\pi}{2}-\alpha=\frac{1}{2}\left(\frac{\pi}{2}-\beta\right),$$

that is, it bisects the angle between the vertical and the plane on which the range is measured.

106. *To find the elevation necessary to the particle's passing through a given point.*

Suppose the point in the axis of x and distant a from the origin. Then we must have

$$\frac{V^2}{g}\sin 2\alpha=a.$$

Let α' be the smallest positive angle whose sine is $\frac{ga}{V^2}$.

The admissible values of α are $\frac{\alpha'}{2}$ and $\frac{\pi-\alpha'}{2}$; so that we see there are two directions in which a particle may be projected so as to reach the given point, and that these are equally inclined to the direction of projection $\left(\alpha=\frac{\pi}{4}\right)$ which gives the greatest range.

Suppose the given point in the plane which makes an angle β with the horizon. Then if its abscissa be a, we must have

$$\frac{2V^2}{g\cos\beta}\cos\alpha\sin(\alpha-\beta)=a.$$

If α', α'' be the two values of α which satisfy this equation, we must have

$$\cos\alpha'\sin(\alpha'-\beta)=\cos\alpha''\sin(\alpha''-\beta);$$

$$\text{and therefore } \alpha''-\beta=\frac{\pi}{2}-\alpha',$$

$$\text{or } \alpha''-\frac{1}{2}\left(\frac{\pi}{2}+\beta\right)=\frac{1}{2}\left(\frac{\pi}{2}+\beta\right)-\alpha'.$$

Hence, as before, the two directions of projection, which enable the particle to strike a point in a given plane through the point of projection, are equally inclined to the direction of projection required for the greatest range along that plane.

107. *To find the envelop of all the trajectories corresponding to different values of α.*

Differentiating equation (3) with respect to α, we get

$$\sec^2\alpha-\frac{gx}{V^2}\frac{\sin\alpha}{\cos^3\alpha}=0,$$

$$\text{or } \tan\alpha=\frac{V^2}{gx}\text{ ....(4).}$$

The elimination of α between (3) and (4) gives us as the equation to the required envelop

$$y=\frac{V^2}{2g}-\frac{gx^2}{2V^2},$$

$$\text{or } x^2=-\frac{2V^2}{g}\left(y-\frac{V^2}{2g}\right).$$

This represents a parabola, whose axis is vertical, whose focus is the point of projection, and whose vertex is in the common directrix of the trajectories.

It will easily be seen from what has gone before that there are two directions of projection, so that the particle may pass through any given point within this parabola, only one for a point in it; and of course there is no possibility of its reaching (with the given velocity V) any point without this parabola.

108. By a somewhat simpler method of considering the problem we might easily have arrived at some of the more obvious properties of the trajectory, thus.

Take the direction of projection as the axis of x, and the vertical downwards from the point of projection as that of y. By the second law of motion we may consider the velocity due to projection to be maintained constant $= V$ parallel to the axis of x, while we have in addition parallel to the axis of y the portion due to gravity as investigated in § 77.

$$\text{Hence } \left.\begin{aligned} x &= Vt \\ y &= \tfrac{1}{2}gt^2 \end{aligned}\right\} \text{ at any time,}$$

$$\text{and therefore } x^2 = \frac{2V^2}{g}\, y;$$

the equation to a parabola referred to a diameter and the tangent at its vertex. The distance of the origin from the directrix, being $\frac{1}{4}^{\text{th}}$ of the coefficient of y, is $\dfrac{V^2}{2g}$, and the velocity due to a fall through that space is as before

$$\sqrt{\left(2g \cdot \frac{V^2}{2g}\right)} = V.$$

109. Many properties of parabolic motion are more easily obtained by geometry than by analysis. We proceed to give a few examples.

Thus suppose O in the figure to be the point of projection, MN the directrix common to the trajectories of all particles projected from O in the plane of the figure with a given velocity, and suppose it be required to determine the direction of projection for the greatest range along the plane OS. Since O is a point in each trajectory and MN the common directrix,

the foci of all possible trajectories lie in the circle $MF'FF''$ described with center O and touching MN in M.

Take any point in this circle as F', then the path whose focus is F' will intersect OS again in a point P' such that if $P'N'$ be drawn perpendicular to MN, $F'P' = P'N'$. Now in order that P' may be as far as possible from O, at P suppose, it is evident (ex absurdo) that the focus must be taken at the point F where OS meets the circle. But the tangent at O bisects the angle between the diameter MO and the focal distance OF. Hence the direction of projection for the greatest range on an inclined plane bisects the angle between the plane and the vertical.

Again, if with center P' and radius $P'F'$ an arc be described cutting $F'FF''$ in F'', it is evident that the trajectories whose foci are F', F'', will intersect OS in the same point P'. Hence, since the directions of projection for these cases will bisect the angles MOF', MOF'' respectively, we see that to strike a given object there are in general two directions of

projection, and that these are equally inclined to the direction which gives the greatest range on the plane passing through the object and the point of projection.

Again, for the envelop of all the trajectories. It is evident that P must be a point in the envelop; since it is the ultimate position of P', when the two parabolas which intersect in that point have become indefinitely nearly coincident. Draw PN perpendicular to MN, and produce it till $NQ = FO$. Draw QR parallel to MN, and cutting OM in R. RQ is a fixed line since $RM = MO$, and as $OP = PQ$ we see that the envelop is a parabola whose focus is O and directrix RQ.

It may be seen at once that it touches in P the only trajectory which can pass through that point. Since the tangent of either curve at P bisects the angle OPQ or FPN.

110. Ex. *It is required to throw a shell with given velocity so as to strike at right angles an inclined plane through the point of projection.*

The letters being the same as before, join ST cutting

$MF'F''$ in F''. Draw $F''P'N'$ perpendicular to MS cutting OS in P'. Find F' so that $P'F' = P'F'' = P'N'$. P' is a point in the trajectory whose focus is F'. Hence the tangent at P'

bisects $F'P'N'$. But OP' bisects $F'P'F''$. Hence the trajectory at P' is perpendicular to OS.

Also as F'' is the focus of the other path by which the point P' might be reached, P' will be the vertex of that path, and therefore the particle will be moving horizontally when it reaches P'.

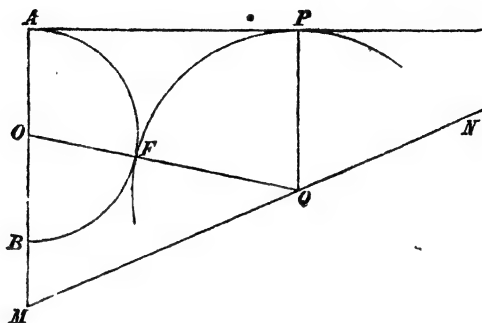
111. Even if the plane along which the range is measured do not pass through the point of projection, a somewhat similar construction will enable us to find the direction of projection for the maximum range. Thus,

Let it be required to find the direction of projection from O with velocity due to AO in order that the range on a horizontal line MN may be a maximum.

Suppose Q' the point where the projectile falls. Join $Q'F''$, $F''O$, F'' being the focus of the path. Then if $Q'P'$ be vertical and meet the horizontal line through A in P', we have $F''Q' = Q'P'$. This is true of each of the paths, and $Q'P'$ is constant. The farthest point Q which can be reached will therefore be determined by inflecting OQ to MN, where $OQ = OA + PQ$, and therefore if $AO = a$, $AM = b$, the cosine of double the requisite angle of elevation will be $\left(\frac{b-a}{b+a}\right)$.

Should MN be an inclined plane, we must evidently draw a line QO, and the corresponding vertical QP; such that if QO meet the circle in F, $FQ = QP$.

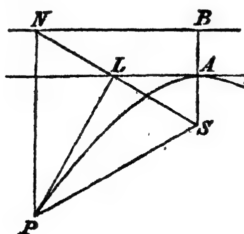
This resolves itself into the well-known geometrical problem of describing a circle whose center is in a given line, and which touches a given circle, and a given straight line.



Of the two solutions, which this problem admits of, one belongs to MN, the other to MN produced to the other side of the point of projection.

112. Perhaps, however, the most satisfactory method of solving all such problems about the maximum range, is to describe the parabola which envelops all the trajectories. The point where this cuts the plane, &c. on which the range is estimated, gives the maximum value of the range, and it is then easy from known properties of the envelop to construct for the required path.

113. Let P be any point in the trajectory, S its focus, BN, AL, the directrix, and the tangent at the vertex.



Then (velocity at $P)^2 = 2g\,PN = 2g\,SP$

$$= \text{(by a property of the parabola) } \frac{2g}{SA}\,SL^2 = \frac{g}{2SA}\,SN^2.$$

Hence velocity at $P \propto SN$; and, since by the figure $SL = LN$, PL is the tangent at P and is perpendicular to SN.

Hence as SN is perpendicular to the direction of motion at P, proportional to the velocity at P, and drawn from a fixed point S, the locus of N is the Hodograph (§ 18) turned through a right angle about S. As this is a horizontal straight line, the Hodograph is a vertical line.

This result will be found of considerable utility in solving various problems in the common vacuum theory of projectiles. It is evident that SB, BN represent the horizontal and vertical velocities at P, in the same scale in which SN represents the entire velocity at that point.

114. *When a particle moves subject to the action of two centers of force where the law is the direct distance and the absolute intensities the same, but one attractive and the other repulsive, its motion will be the same as that of a projectile in vacuo.*

For the whole force on the particle resolved perpendicular to the line joining the centers is evidently zero, and that parallel to this line is equal to that which would be exerted by either of the centers on a particle placed at the other; and always tends in the direction parallel to that from the repelling, to the attracting, center. It corresponds therefore exactly to the force of gravity, within moderate elevations above the earth's surface.

115. Again if a particle moves on a plane inclined to the horizon at an angle θ, the whole force on it is, by § 79, $g \sin\theta$ parallel to the line of greatest slope on the plane, and therefore the trajectory will still be a parabola, whose dimensions will depend upon θ.

Ex. *A particle is projected from a given point with a given velocity, and moves on an inclined plane; find the locus of the directrices of its path for different inclinations of the plane.*

It will be easily seen that when a particle moves on an inclined plane, the velocity at any point is equal to that which would have been acquired by sliding from the directrix; that is (§ 80) equal to the velocity due to the fall from a horizontal plane through the directrix. Now the velocity is given constant, hence the locus of the directrices is a horizontal plane.

116. *A particle moves subject to the action of a force always perpendicular to a given plane, and a function of the distance of the particle from the plane: to determine the motion.*

It is evident that the motion will be confined entirely to a plane through the direction of projection perpendicular to the attracting plane. Let us take the former as the plane of xy, the axis of x lying in the attracting plane. Let $\phi'(D)$ be the acceleration at distance D, where ϕ' is the derived function of ϕ. Then the equations of motion are

$$\frac{d^2x}{dt^2}=0, \quad \frac{d^2y}{dt^2}=-\phi'(y).$$

Suppose the particle projected from a point (a, b), in a direction making an angle α with the axis of x, and with a velocity V.

Multiplying by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$, and integrating we get

$$\left.\begin{aligned}\left(\frac{dx}{dt}\right)^2&=\text{const.}=V^2\cos^2\alpha,\\ \left(\frac{dy}{dt}\right)^2&=C-2\phi(y)=V^2\sin^2\alpha+2\phi(b)-2\phi(y).\end{aligned}\right\}\text{......(1).}$$

Hence $v^2=V^2+2\{\phi(b)-\phi(y)\}$ and therefore depends only on the distance from the attracting plane, a particular case of conservation of energy.

To find the differential equation to the path, we have

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{dy}{dx}=\frac{\sqrt{[V^2\sin^2\alpha+2\{\phi(b)-\phi(y)\}]}}{V\cos\alpha},$$

an equation integrable for particular forms only of the function ϕ. An interesting case is that in which the attraction of the plane is inversely as the cube of the distance,

$$\text{or } \phi'(y) = \frac{\mu}{y^3}, \text{ and therefore } \phi(y) = -\tfrac{1}{2}\frac{\mu}{y^2}.$$

The differential equation becomes

$$\frac{dy}{dx} = \frac{\sqrt{\left\{\left(V^2 \sin^2 \alpha - \frac{\mu}{b^2}\right) + \frac{\mu}{y^2}\right\}}}{V \cos \alpha}.$$

There will be three cases according as $\left(V^2 \sin^2 \alpha - \frac{\mu}{b^2}\right)$ is positive, zero or negative.

1st. Let it be positive and $= \frac{\mu}{a_1^2}$,

$$y\frac{dy}{dx} = \frac{\sqrt{\mu}}{Va_1 \cos \alpha} \sqrt{(a_1^2 + y^2)};$$

$$\text{whence } \sqrt{(a_1^2 + y^2)} = \frac{\sqrt{\mu}}{Va_1 \cos \alpha}(x + C),$$

the equation of a hyperbola whose *transverse* axis is that of x.

2nd. Let $V^2 \sin^2 \alpha - \frac{\mu}{b^2} = 0$,

$$y\frac{dy}{dx} = \frac{\sqrt{\mu}}{V \cos \alpha},$$

$$\text{and } y^2 = \frac{2\sqrt{\mu}}{V \cos \alpha}(x + C),$$

a parabola whose axis is that of x.

3rd. Let $V^2 \sin^2 \alpha - \frac{\mu}{b^2}$ be negative, and $= -\frac{\mu}{a_1^2}$,

$$y\frac{dy}{dx} = \frac{\sqrt{\mu}}{Va_1 \cos \alpha} \sqrt{(a_1^2 - y^2)},$$

$$\text{or } \sqrt{(a_1^2 - y^2)} = \frac{\sqrt{\mu}}{Va_1 \cos \alpha}(x + C),$$

the equation of an ellipse of which the axis of x is an axis.

We might have obtained the above results by integrating separately the two equations of motion, and then eliminating t between them.

If the force be repulsive, instead of attractive, it is easy to see, by a slight modification of the above process, that there is only one case, and that the curve described is a hyperbola whose *conjugate* axis lies in the intersection of the plane of projection and the attracting plane.

From this we see that the conic sections are the only curves which can be described by a free particle moving in a plane and subject to a force in the direction, and inversely as the cube, of the perpendicular on a given line in that plane.

The converse of either of the above propositions is easily investigated; thus, taking the first, our problem becomes

117. *To find the law of force perpendicular to an axis that a free particle may describe a conic section.*

Take the axis as that of x, and the vertex as origin, then the equation

$$y^2 = 2mx + nx^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

will represent, by properly taking m and n, any parabola, any hyperbola referred to its transverse axis, or any ellipse referred to either axis.

Also, since the force is perpendicular to the axis, we have

$$\frac{dx}{dt} = c.$$

Hence
$$y\frac{dy}{dt} = mc + nxc;$$

$$\text{and } y\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 = nc^2.$$

From these $$\frac{d^2y}{dt^2} = \frac{1}{y}\left\{nc^2 - \left(\frac{dy}{dt}\right)^2\right\}$$

$$= \frac{1}{y}\left\{nc^2 - \frac{(m+nx)^2}{y^2}c^2\right\}$$

$$= \frac{c^2}{y^3}(ny^2 - m^2 - 2mnx - n^2x^2)$$

$$= -\frac{c^2m^2}{y^3} \text{ by equation (1).}$$

For the second case, a hyperbola referred to its conjugate axis taken as that of x, the equation is

$$y^2 = p^2x^2 + q^2.$$

Hence $$y\frac{dy}{dt} = p^2x\frac{dx}{dt}$$

$$= p^2cx,$$

from which we have immediately

$$y\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 = p^2c^2.$$

That is, $$\frac{d^2y}{dt^2} = \frac{1}{y}\left\{p^2c^2 - \left(\frac{dy}{dt}\right)^2\right\}$$

$$= \frac{p^2c^2}{y}\left(1 - \frac{p^2x^2}{y^2}\right)$$

$$= \frac{p^2q^2c^2}{y^3}.$$

118. *To find the force which must act perpendicular to a plane, in terms of the distance from that plane, that a given path may be described.*

Take the axes as before; then, Y being the required force (a function of y only), we have

$$\frac{d^2x}{dt^2} = 0, \text{ or } \frac{dx}{dt} = \text{const.} = a, \text{ suppose;}$$

$$\frac{d^2y}{dt^2} = Y \text{ .... } (1).$$

Let $y = f(x)$ be the equation to the given curve,

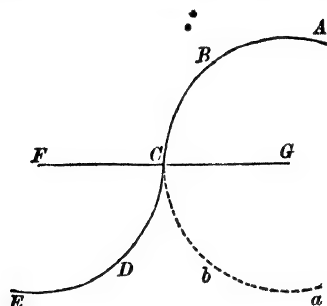
$$\frac{dy}{dt} = af'(x),$$

$$\frac{d^2y}{dt^2} = a^2f''(x),$$

$$\text{or by (1),} \quad Y = a^2f''(x)$$
$$= a^2f''\{f^{-1}(y)\},$$

by the equation to the curve. Hence the law of force required is found.

119. It is necessary to observe that, in the case of § 116, when the particle actually reaches the axis, it will not proceed to describe the portion of the same curve which lies on the



other side of the axis, as this would involve a change in sign of the constant horizontal velocity. It is, in fact, evident that in such cases the particle having described ABC will, instead of pursuing the course Cba, actually describe CDE similar and equal to Cba, but turned in the opposite direction. And a similar remark applies to the general problem in § 118.

Although, in the case of ABC being a conic, one of whose axes is CG, and therefore cutting it at right angles in C, it might seem that at C the horizontal velocity vanishes, yet it is to be recollected that the velocity at C is infinitely great;

and it may easily be shewn by independent methods, such as limits, if the foregoing analysis do not appear satisfactory, that the velocity parallel to CG is really constant throughout the motion.

120. It may be useful to notice that cases of this kind are reduced at once to investigations similar to those of last Chapter, by considering, separately, the equations of motion parallel and perpendicular to the attracting plane.

Whenever, then, we can completely determine the motion of a particle towards a center of force, in a straight line, we can also completely solve the problem of the motion of a particle anyhow projected, and attracted by an infinite plane; the law of force in terms of the distance being the same in the two cases.

121. *Generally, when a particle is anyhow projected and subject only to the action of a force whose direction is perpendicular to a given plane, and whose magnitude depends solely on the distance from the plane; the velocity parallel to that plane is constant; and, in passing from any point to another, the square of the entire velocity is altered by a quantity depending only upon the distances of these two points from the given plane.*

Take the axis of y perpendicular to the given plane, and the axis of x in it, so that the direction of projection lies in xy. This will evidently be the plane of motion; and the equations are

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = Y.$$

Hence $$\frac{dx}{dt} = c,$$

and $$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = V^2 + 2\int_{y_1}^{y} Ydy$$

$$= V^2 + \phi(y, y_1),$$

V being the velocity of projection, and y_1 the co-ordinate of the point of projection; which proves the proposition.

This is, of course, merely a particular case of the general principle of Conservation of Energy (§ 73).

122. As another example of the motion of a particle under the action of forces whose direction is constant, *let us consider the motion of a particle of light in the corpuscular theory, at the confines of two homogeneous media whose bounding surface is plane.*

In this case the hypothesis is that the attractive or repulsive forces, exerted by the particles of any medium on a particle of light passing through it, are insensible at sensible distances but enormously great at infinitely small distances. Hence of course the path of such a particle in a homogeneous medium will be a straight line, and will be described with constant velocity, until the particle is infinitely near to the bounding surface of the medium.

Thus, suppose AB to be the common plane surface of two such media. Draw CD at a distance from AB equal to that at which the intensity of the attractive forces of the particles of the medium begins to be sensible; and draw EF parallel to CD and equidistant from it with AB. By what we have just noticed, a particle of light moving along PQ will arrive at Q without any change of velocity or direction. Also from the symmetry of the figure, the resultant of all the sensible forces on it will always be perpendicular to AB. This shews, § 121, that the velocity resolved parallel to AB is constant throughout the motion, and also that whatever be the direction of PQ, the change in the square of the velocity in passing from Q to any point of the path will depend only on the distance of that point from AB.

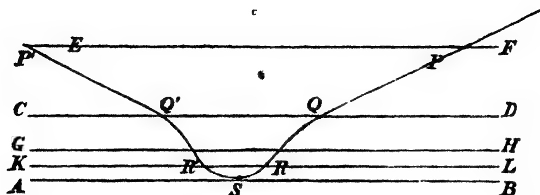
Let PQR represent a portion of the path.

We have no means of determining its actual form, since the extent through which the force is sensible, the law of its variation, and whether it change from attractive to repulsive with the distance, are unknown.

Through any point R draw KRL parallel to AB, and let GH be equidistant from KL with AB.

Then at R the particle is subject only to the actions of the upper medium beyond GH, and of the lower medium.

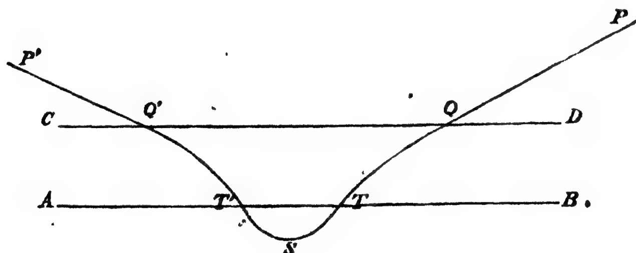
If the resultant effects of these two should, at a point S in the superior medium, destroy the velocity perpendicular to AB, the particle will evidently pursue a course $SR'Q'P'$



similar and equal to $SRQP$, and the angles $P'Q'C$ and PQD will be equal, as also the velocities in PQ and $P'Q'$. (§ 121.) *Here we have the case of a ray reflected at a plane surface.*

If however the attraction of the lower medium should so prevail that the particle actually enters it, then we may consider its motion, while still within the range of action of both media precisely as before; but there will be two cases.

I. At some point as S whose distance from AB (the bounding surface) is less than that of AB from CD, the velocity perpendicular to AB may be destroyed; then as before,



the particle will pursue the path $ST'Q'P'$, similar and equal to $STQP$, and will be reflected at an angle equal to that of incidence and with its original velocity.

II. The particle may pass into the lower medium so far as to be independent of the action of the upper medium.

After this it will move in a straight line as before, and the change of the square of its velocity will be, § 121, independent of the path pursued. Hence, if V be the velocity, and α the angle, of incidence; V', α' those of refraction, we have

$$V \sin \alpha = V' \sin \alpha',$$

$$V^2 = V'^2 \pm a^2,$$

where a is a constant depending on the nature of the two media.

Hence, $$\frac{\sin \alpha}{\sin \alpha'} = \frac{V'}{V} = \sqrt{\left(1 \mp \frac{a^2}{V^2}\right)},$$

and, therefore, for particles of light which have the same velocity the ratio of the sines of the angles of incidence and refraction is constant. *This is the known law of ordinary refraction.*

We have introduced this example, although belonging to a theory now completely exploded, as it forms a good illustration of the application of the results of this Chapter, and was the first instance of the solution of a problem connected with molecular forces. It is due to Newton.

EXAMPLES.

(1) The time of describing any portion PQ of the parabolic path of a particle acted on by gravity, is proportional to the difference of the tangents of the angles which the tangents at P and Q make with the horizon. (§ 113.)

(2) The sights of a gun are set so that the ball may strike a given object; shew that when the sights are directed to any other object in the same vertical line, the ball will also strike it.

(3) Shew that the time of a projectile's describing an arc of its path cut off by a focal chord is equal to the time of falling vertically from rest through a space equal to the chord.

(4) If a shell burst, all the fragments receiving equal

velocities from the explosion; shew that the locus of the foci of the paths of the fragments is a sphere, of the vertices an oblate spheroid, and of the particles themselves at any time a sphere.

(5). Two bodies, projected from the same point A, in directions making angles α, α' with the vertical, pass through the point B in the horizontal plane through A; prove that if t, t' be the times of flight from A to B,

$$\frac{\sin(\alpha-\alpha')}{\sin(\alpha+\alpha')}=\frac{t'^2-t^2}{t'^2+t^2}.$$

(6) With what velocity must a projectile be fired at an elevation of 30°, so as to strike an object at the distance of 2500 feet on an ascent of 1 in 40?

(7) ABC is a right-angled triangle in a vertical plane with its hypothenuse AB horizontal; a particle projected from A passes through C and falls at B; prove that the tangent of the angle of projection $= 2 \operatorname{cosec} 2A$, and that the latus rectum of the path described is equal to the height of the triangle.

(8) If a body be projected at an angle α to the horizon with the velocity due to gravity in 1^s, its direction is inclined at an angle $\frac{\alpha}{2}$ to the horizon at the time $\tan\frac{\alpha}{2}$, and at an angle $\frac{\pi-\alpha}{2}$ at the time $\cot\frac{\alpha}{2}$.

(9) A plane AB inclined at an angle α to the horizon, leads up to a horizontal plane BC; a particle is projected with a velocity V from the point A, traverses the plane AB, and falls upon the horizontal plane BC; if the times of motion from A to B and from B to C be equal, shew that

$$AB=\frac{2V^2}{g}\,\frac{\sin\alpha\,(1+\sin^2\alpha)}{(1+2\sin^2\alpha)^2}.$$

(10) Three particles are projected simultaneously from the same point, and strike the horizontal plane through the point simultaneously; prove that, if their ranges be in geo-

metrical progression, the latera recta of their paths will also be in geometrical progression.

(11) If u and v be the velocities at the extremities of a focal chord of a projectile's path, V_x the horizontal velocity, shew that

$$\frac{1}{u^2}+\frac{1}{v^2}=\frac{1}{V_x^2}. \quad (\S\ 113).$$

(12) From a point in an inclined plane two bodies are projected with the same velocity in the same vertical plane in directions at right angles to each other; the difference of their ranges is constant.

(13) A ball is thrown up in a vertical plane passing through the sun, in a direction inclined at an angle θ to the horizon, and it is observed that t seconds elapse from the instant that the ball is in the line joining the point of projection with the sun till it reaches the ground again, and that T seconds is the whole time of flight: shew that

$$t \tan \theta = T \tan \alpha,$$

where α is the altitude of the Sun.

(14) Find an expression for the velocity of the shadow on the ground in (13); and shew that its greatest distance from the point of projection is $\dfrac{V^2 \sin^2 (\theta - \alpha)}{g \sin 2\alpha}$, and that it will attain this position after a time $\dfrac{V \sin (\theta - \alpha)}{g \cos \alpha}$, V being the velocity of projection. Prove also that the shadow moves with a uniform acceleration $g \cot \alpha$.

(15) A particle is projected from the top of a tower with the velocity which would be acquired in falling vertically down n times the height of the tower, find the range on the horizontal plane through the bottom of the tower, and shew that it will be a maximum when the angle of projection is

$$\frac{1}{2} \sec^{-1} (1 + 2n).$$

(16) Two inclined planes of equal altitudes h, and inclined at the same angle α to the horizon, are placed back to back upon a horizontal plane. A ball is projected from the foot of one plane along its surface and in a direction making an angle β with its line of intersection with the horizontal plane. After flying over the top of the ridge it falls at the foot of the other plane; shew that the velocity of projection is

$$\frac{1}{2}\sqrt{gh}\operatorname{cosec}\beta\sqrt{8+\operatorname{cosec}^2\alpha}.$$

(17) Two bodies A, B acted on by gravity are projected from two given points in the same vertical line with the same velocity and in parallel directions; shew that if A be higher than B, pairs of tangents drawn to B's path from any points of A's path will intercept arcs described by B in equal times.

(18) If v, v', v'', be the velocities at three points P, Q, R, of the path of a projectile where the inclinations to the horizon are α, $\alpha-\beta$, $\alpha-2\beta$; and if t, t' be the times of describing PQ, QR respectively, shew that

$$v''t = vt', \text{ and } \frac{1}{v}+\frac{1}{v''}=\frac{2\cdot\cos\beta}{v'}. \quad (§ 113).$$

(19) If two particles be projected from the same point at the same instant in the same vertical plane, with velocities v and v_1 in directions making angles α and α_1 with the horizon; shew that the interval between their transits through the other point which is common to their paths is

$$\frac{2}{g}\cdot\frac{vv_1\sin(\alpha-\alpha_1)}{v_1\cos\alpha_1+v\cos\alpha}.$$

(20) If any chord be drawn to the trajectory of a projectile the velocities of the particle at its extremities if resolved perpendicular to the chord, are equal. (§ 113).

(21) Particles slide from rest at the highest point of a vertical circle down chords, and are then allowed to move freely; shew that the locus of the foci of their paths is a circle

of half the dimensions, and that all the paths bisect the vertical radius.

(22) If the particles slide down chords to the lowest point, and be then suffered to move freely, the locus of the foci is a cardioide.

(23) Down what chord from the vertex of a vertical circle must a particle slide so as to have when falling freely the greatest range on a given horizontal plane?

(24) Find the locus of the foci of all trajectories which pass through two given points.

(25) The envelop of all the parabolas which correspond to a given velocity of projection is equal to the trajectory for which the direction of projection is horizontal.

(26) Particles fall down *diameters* of a vertical circle; the locus of the foci of their subsequent paths is the circle.

(27) If two bodies be projected from the same point, with equal velocities, and in such directions that they both arrive at the same point of a plane whose inclination to the horizon is β, and if t, t' be the times of flight, and α the angle of projection of the first,

$$t' = t \cdot \frac{\cos \alpha}{\sin (\alpha - \beta)}.$$

(28) If the focus of the projectile's path be as much below the horizontal plane through the point of projection, as the vertex is above; shew that double the angle of projection

$$= \sec^{-1} 3.$$

(29) From points of an inclined plane, particles are simultaneously projected in different directions. If their times of flight are the same, shew that their locus at any instant is a plane parallel to the given one.

(30) A particle is thrown over a triangle from one end of the horizontal base, and, grazing the vertex, falls upon the

other end of the base. If α, β be the base angles, θ the angle of projection,

$$\tan\theta = \tan\alpha + \tan\beta.$$

(31) For the greatest range on an inclined plane through the point of projection the direction of motion on leaving, is at right angles to that on reaching, the plane.

(32) Particles are projected from the same point in a vertical plane: 1st, with the same vertical, 2nd, with the same horizontal, velocity; shew that in each case the locus of the foci is a parabola whose focus is at the point of projection, and axis vertical, but whose vertex is upwards in case (1) and downwards in (2).

(33) If α be the angle of projection, T the time which elapses before the projectile strikes the ground, prove that at the time $\dfrac{T}{4\sin^2\alpha}$ the angle which the direction of motion makes with the direction of projection is $\dfrac{\pi}{2} - \alpha$.

(34) If a body describe an arc of a cycloid under the action of a force parallel to the base, shew that this force varies inversely as $2\sin\theta - \sin 2\theta$, θ being the corresponding arc of the generating circle measured from the vertex.

(35) If the force perpendicular to a plane vary as the distance, shew that the curves described have equations of the form

$$\left.\begin{array}{l} y = Aa^x + Ba^{-x}, \\ \text{or } y = A\cos(mx + B) \end{array}\right\} \begin{array}{l}\text{as the force is repulsive} \\ \text{or attractive.}\end{array}$$

Find the circumstances of projection in the two cases that the curves may be the catenary, and the companion to the cycloid, respectively.

(36) Particles are projected in the same plane and from the same point, in such a manner that the parabolas described are equal; prove that the locus of the vertices of these parabolas will be a parabola.

CHAPTER V.

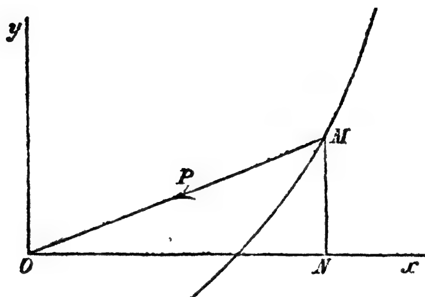
CENTRAL FORCES.

123. In this part of the subject we consider the motion of a particle under the action of a force whose direction always passes through, and whose intensity is some function of the distance from, a fixed point. The fixed point is called the *Center of Force*, and the force is said to be attractive or repulsive according as it is directed to or from the center. The former, as including the most important applications of the subject, we will take as our standard case; but it will be seen that a simple change of sign will adapt our general formulæ to the latter. If the center of force be itself in motion, the methods of §§ 24, 28, enable us easily to treat it as fixed; but in this case the relative acceleration is not in general directed to the center, so that the problem no longer belongs to *Central Forces* strictly so called. It will be considered later. If the center be moving uniformly in a straight line, the results of this chapter are at once applicable to the relative motion.

124. *A particle is projected in a plane, and is acted on by a force P directed to the fixed point O in that plane; to determine the motion.*

The whole motion will clearly take place in the plane, as there is no force to withdraw the particle from it. Let Ox, Oy, any two lines through O at right angles to each other, be taken as the axes of co-ordinates. Let M be the position of the particle at the time t; and draw MN perpendicular to Ox, and join MO. Let $ON=x$, $NM=y$, $OM=r$, and the angle $NOM=\theta$. Then, since $\cos\theta=\frac{x}{r}$, $\sin\theta=\frac{y}{r}$, the components of P, parallel to the axes and in the negative di-

rections, are $P\frac{x}{r}$, $P\frac{y}{r}$. But by the second law of motion



we may consider the accelerations in the directions of x and y separately, and we have therefore

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=-P\frac{x}{r}\\ \frac{d^2y}{dt^2}&=-P\frac{y}{r}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(A).$$

In these, since P is a function of r, and therefore of x and y, the second members will generally contain both these variables, and the equations must be treated as simultaneous differential equations. Their integrals will give x, y, $\frac{dx}{dt}$, $\frac{dy}{dt}$, in terms of t; from which the position and velocity of the particle at any instant will be known, and the problem completely solved. In one case, however, viz. when P is proportional to r, the first equation will involve x and t, and the second y and t, only, and each equation may be integrated by itself. As it is the simplest example of its class, and of great importance in its applications, especially to Acoustics and to Physical Optics, we will begin by considering it.

125. *A particle moves about a center of force, the force varying directly as the distance: to determine the motion.*

Let μ be the acceleration at unit of distance, usually called

the absolute force of the center, then $P = \mu r$, and equations (A) become

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -\mu x \\ \frac{d^2y}{dt^2} &= -\mu y \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (B),$$

the integrals of which, see § 83, are

$$x = A \cos \{\sqrt{\mu}t + B\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$y = A' \cos \{\sqrt{\mu}t + B'\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

A, B, A', B' being the constants introduced in the integration, to be determined by the initial circumstances of motion. Consider the particle projected from a point on the axis of x, at distance a from the center, with velocity V, and in a direction making an angle α with Ox. When $t = 0$, we have $x = a$, $y = 0$, $\frac{dx}{dt} = V \cos \alpha$, $\frac{dy}{dt} = V \sin \alpha$. Hence,

$$a = A \cos B,$$

$$0 = A' \cos B',$$

$$V \cos \alpha = -A\sqrt{\mu} \sin B,$$

$$V \sin \alpha = -A'\sqrt{\mu} \sin B'.$$

Expanding the cosines in (1) and (2), and substituting these expressions for the constants, we obtain

$$x = \frac{V \cos \alpha}{\sqrt{\mu}} \sin \sqrt{\mu}t + a \cos \sqrt{\mu}t \quad \ldots\ldots\ldots\ldots (3),$$

$$y = \frac{V \sin \alpha}{\sqrt{\mu}} \sin \sqrt{\mu}t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

which contain the complete solution of the problem. Eliminating t, we have

$$(x \sin \alpha - y \cos \alpha)^2 + \frac{\mu a^2}{V^2} y^2 = a^2 \sin^2 \alpha,$$

the equation to the path of the particle; which is therefore an ellipse whose center is O. Equations (3) and (4) give periodic values for x, y, $\frac{dx}{dt}$, $\frac{dy}{dt}$, such that all the circumstances of motion will be the same at the time $t+\frac{2\pi}{\sqrt{\mu}}$ as at the time t. The period of revolution is therefore $\frac{2\pi}{\sqrt{\mu}}$: a most remarkable result, as it is independent of the dimensions of the ellipse, and depends solely on the intensity of the force.

By taking μ negative in equations (B), we may apply them to the case of a repulsive force varying as the distance from O. In the integration for this supposition the sines and cosines would be replaced by exponentials, and the curve described would be a hyperbola having O as center; but the motion would not be one of revolution, as the particle would necessarily always remain on the same branch of the hyperbola.

126. Recurring to equations (A), it will in all cases but the one we have just considered be more convenient to transform them to polar co-ordinates, especially as the general polar differential equation to the orbit described by a particle under the action of a central force can be easily formed, as follows.

127. *A particle being acted on by a central force; it is required to determine the polar equation to the path.*

Multiplying the second of equations (A), § 124, by x, and the first by y, and subtracting, we obtain

$$x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=0.$$

Integrating,

$$x\frac{dy}{dt}-y\frac{dx}{dt}=\text{constant}=h \text{ suppose.}$$

Changing the variables from x, y, to r, θ, where $x=r\cos\theta$, $y=r\sin\theta$, we get as in § 22,

$$r^2 \frac{d\theta}{dt} = h \text{..........................} (1),$$

or, substituting $\frac{1}{u}$ for r,

$$\frac{d\theta}{dt} = hu^2 \text{..........................} (2).$$

Again, $x = r \cos\theta = \frac{\cos\theta}{u}$;

which gives $\frac{dx}{dt} = -\frac{u \sin\theta + \cos\theta \frac{du}{d\theta}}{u^2} \frac{d\theta}{dt}$

$$= -h\left(u \sin\theta + \cos\theta \frac{du}{d\theta}\right), \text{ by (2)};$$

and therefore $\frac{d^2x}{dt^2} = -h\left(u \cos\theta + \cos\theta \frac{d^2u}{d\theta^2}\right) \frac{d\theta}{dt}$

$$= -h^2u^2\left(u \cos\theta + \cos\theta \frac{d^2u}{d\theta^2}\right), \text{ by (2)}.$$

But, by the first of equations (A),

$$\frac{d^2x}{dt^2} = -P \cos\theta.$$

Equating these values of $\frac{d^2x}{dt^2}$, and dividing by $\cos\theta$, we have

$$P = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) \text{..................} (3),$$

$$\text{or } \frac{d^2u}{d\theta^2} + u - \frac{P}{h^2u^2} = 0 \text{................} (4).$$

This is the differential equation to the orbit described; and as, in any particular instance, P will be given in terms of r, and therefore in terms of u, its integral will be the polar equation to the required path.

128. The general integrals of (A), which are differential equations of the second order, ought to contain four constants. One of these has been already introduced in (1), and two more will be introduced by the integration of (4). If the value of r deduced from the integral of (4) be substituted in (1), and that equation be then integrated, the remaining constant will be introduced, and the path of the particle and its position at any time will be obtained. The four constants involved in the resulting equations must be determined from the initial circumstances of motion; namely, the initial position of the particle (depending on *two* independent co-ordinates), its initial velocity, and its direction of projection.

129. Equation (3) may be used to ascertain the law of central force which must act upon a particle to cause it to describe a given curve. To effect this we must determine the relation between u and θ from the polar equation to the orbit referred to the required center of force as pole; we must then differentiate u twice with respect to θ, and substitute the result in the expression for P; eliminating θ, if it be involved, by means of the relation between u and θ. In this way we shall obtain P in terms of u alone, and therefore of r alone.

When we know the relation between r and θ from (4), we make use of equation (1) to determine the time of describing a given portion of the orbit; or, conversely, to find the position of the particle in its orbit at any time.

130. The equation of the orbit between r and p, the radius vector and the perpendicular on the tangent at any point, may be easily obtained from (4). For by *Diff. Calc.* we have

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{p^3u^2}\frac{dp}{dr};$$

$$\text{and therefore } P = \frac{h^2}{p^3}\frac{dp}{dr}.$$

131. *The sectorial area swept out by the radius vector of the particle in any time is proportional to the time* (§ 22).

$\frac{1}{h^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$, differentiating [illegible]

If A denote this area we have, by *Diff. Calc.*

$$\frac{dA}{dt}=\frac{r^2}{2}\frac{d\theta}{dt};$$

and therefore, by equation (1) of § 127,

$$\frac{dA}{dt}=\frac{h}{2},$$

$$\text{whence } A=\tfrac{1}{2}ht+C=\tfrac{1}{2}ht,$$

C being zero if A and t be supposed to vanish together. Let A' be the area described in any other interval t', then

$$A'=\tfrac{1}{2}ht';$$

$$\text{and therefore } A : A' :: t : t';$$

or, the areas described in different intervals are proportional to these intervals. We also see, by taking $t=1$, that the value of h is twice the area described in a unit of time.

132. *The velocity of the particle at each point of its path is inversely proportional to the perpendicular from the center of force on the tangent at that point.* (§ 21.)

For $$\text{Velocity}=v=\frac{ds}{dt}$$

$$=\frac{ds}{d\theta}\frac{d\theta}{dt}$$

$$=\frac{r^2}{p}\frac{d\theta}{dt}, \text{ by } \textit{Diff. Calc.}$$

(p being the perpendicular on the tangent from the center of force)

$$=\frac{h}{p}, \text{ by equation (1) of § 127.}$$

Hence, as above, $v \propto \dfrac{1}{p}$.

133. This equation enables us to express h in terms of the initial circumstances of the motion. For, let R be the distance of the point of projection from the center, V the velocity, and β the angle which the direction of projection makes with that of R. Then evidently

Perpendicular on tangent at point of projection $= R \sin \beta$;

$$\text{or } V = \frac{h}{R \sin \beta},$$

$$\text{whence } h = VR \sin \beta.$$

Again, since by *Diff. Calc.*,

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2,$$

we have

$$v^2 = \frac{h^2}{p^2} = h^2 \left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\},$$

another important expression for the velocity.

134. *The velocity at any point of a central orbit is independent of the path described, and depends solely on the intensity and law of the force; the distance of the point from the center, and the velocity and distance of projection.*

Multiply equations (A) § 124, by $\frac{dx}{dt}$, $\frac{dy}{dt}$ respectively, and add, then

$$\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} = -\frac{P}{r}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= -P\frac{dr}{dt}.$$

$$\left(\text{Since } x^2 + y^2 = r^2, \text{ we have } x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}\right).$$

$$\text{But } v^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2;$$

Hence $\frac{d(v^2)}{dt} = -2P\frac{dr}{dt}$.

Also, since P is some function of r, let $P = \phi'(r)$, then

$$\tfrac{1}{2}v^2 = C - \int \phi'(r)\,dr$$
$$= C - \phi(r).$$

At the point of projection $v = V$, $r = R$; and therefore

$$\tfrac{1}{2}V^2 = C - \phi(R);$$
$$\text{whence } \tfrac{1}{2}v^2 - \tfrac{1}{2}V^2 = \phi(R) - \phi(r),$$

which proves the proposition. (Compare § 73).

135. *The velocity of a particle at any point of a central orbit is the same as that which would be acquired by a particle moving freely from rest along one-fourth of the chord of curvature at the point, drawn through the center of force, under the action of a constant force whose intensity is equal to that of the central force at the point.*

By § 134,

$$\frac{d(v^2)}{dt} = -2P\frac{dr}{dt};$$

$$\text{or } v\frac{dv}{dr} = -P.$$

And by § 132,

$$v = \frac{h}{p}.$$

Differentiating the logarithm of the latter, we obtain

$$\frac{1}{v}\frac{dv}{dr} = -\frac{1}{p}\frac{dp}{dr},$$

and, dividing the former equation by this,

$$v^2 = Pp\frac{dr}{dp} = 2P\frac{p}{r}\cdot\frac{2r}{4}\frac{dr}{dp}$$

$$= 2P\frac{q}{4}, \text{ (Diff. Calc.)}$$

where q is the chord of curvature through the center. Hence the proposition, § 77.

From this it follows that the velocity, V, of a particle moving in a circle of radius R, under the action of any force P to the center, is given by the equation

$$V^2 = PR,$$

a simple, and most useful expression.

136. Def. An *Apse* is a point in a central orbit at which the radius vector is a maximum or minimum, and the corresponding value of the radius vector is called an *Apsidal Distance*.

The analytical conditions for such a point (*Diff. Calc.*) are that $\frac{du}{d\theta}$ should vanish, and that the first succeeding differential coefficient which does not vanish should be of an even order. The first condition ensures that the tangent at an apse is perpendicular to the radius vector.

Every apsidal line divides the orbit into two parts which are equal and similar.

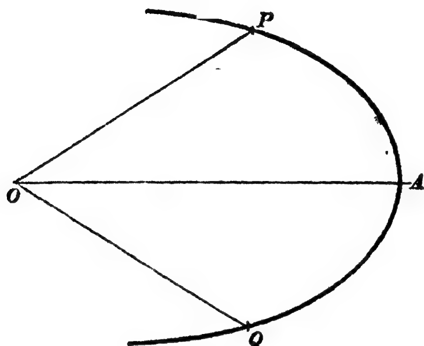
For $\frac{du}{d\theta}$ changes sign in passing through an apse, and therefore, since $\left(\frac{du}{d\theta}\right)^2$ is some function of u, $f(u)$, suppose, if on one side of the apse $\frac{du}{d\theta}$ be represented by $+\sqrt{f(u)}$, on the other it will be represented by $-\sqrt{f(u)}$.

Hence if A be an apse, O the center of force, and OP, OQ any two lines on opposite sides of OA and equally inclined to it, we have

$$\angle QOA = \int_{\frac{1}{OQ}}^{\frac{1}{OA}} \frac{du}{\sqrt{f(u)}},$$

$$\angle AOP = -\int_{\frac{1}{OA}}^{\frac{1}{OP}} \frac{du}{\sqrt{f(u)}}.$$

(*Note.* These integrals have no meaning if there be an apse in AP or in AQ, for then they contain an infinite element.)



$$\text{Hence} \int_{\frac{1}{OQ}}^{\frac{1}{OA}} \frac{du}{\sqrt{f(u)}} + \int_{\frac{1}{OA}}^{\frac{1}{OP}} \frac{du}{\sqrt{f(u)}} = 0,$$

$$\text{or} \int_{\frac{1}{OQ}}^{\frac{1}{OP}} \frac{du}{\sqrt{f(u)}} = 0 \text{ identically.}$$

Whence, if P and Q have been taken so close to A, that no apse but A lies between them, which can of course always be done, unless the orbit is a circle about O, we have

$$OP = OQ,$$

which shews that any two lines on opposite sides of, and equally inclined to, OA are equal. Hence the parts AP, AQ, into which OA divides the orbit, are equal and similar, so long as neither contains an apse. But if P be an apse, it is evident that Q will also be one; and therefore, the portion of the orbit between P and the next apse being similar and equal to PA, and the same being true for Q; these new portions are symmetrical about OA, and so on: and the proposition is completely proved.

137. *In a central orbit there cannot be more than two apsidal distances.*

For, since the parts of the orbit on opposite sides of an apse are similar, the particle after passing two apses must come next to one at an equal distance with that of the first, then to one at an equal distance with that of the second, and so on. Hence there can be but two apsidal distances.

138. When the central force varies as a power of the distance, we may obtain the above result, as well as the equation for determining the apsidal distances, directly from equation (4) of § 127. Suppose $P = \mu u^n$, then we have

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2} u^{n-2} = 0.$$

Multiplying by $2h^2 \frac{du}{d\theta}$ and integrating, we have

$$h^2 \left\{ \left(\frac{du}{d\theta}\right)^2 + u^2 \right\} = v^2 = \frac{2\mu}{n-1} u^{n-1} + C.$$

Suppose the particle projected with a velocity equal to q times the velocity from infinity at the same distance, and let c be the initial value of u, then when $u = c$,

$$v^2 = \frac{2\mu q^2}{n-1} c^{n-1} \text{ (§ 96)};$$

$$\text{whence } C = (q^2 - 1) \frac{2\mu}{n-1} c^{n-1};$$

and therefore $h^2 \left\{ \left(\frac{du}{d\theta}\right)^2 + u^2 \right\} = \frac{2\mu}{n-1} \{u^{n-1} + (q^2 - 1) c^{n-1}\}.$

To determine the apsidal distances we must put $\frac{du}{d\theta} = 0$, which gives

$$u^{n-1} - \frac{h^2 (n-1)}{2\mu} u^2 + (q^2 - 1) c^{n-1} = 0.$$

The form of this equation shews that it can have at mos

two positive roots, which are therefore the two apsidal distances.

Although there can only be two apsidal distances, there may be any number of apses, and the angle between two consecutive apsidal distances is called the apsidal angle. Generally, to determine this angle, the equation to the orbit must first be found for the particular case considered; but the apsidal angle may be determined approximately for any law of force, without first finding the form of the orbit, if we assume that it does not differ much from a circle.

139. *A particle revolves in an orbit which is very nearly circular, and is acted on by a central force varying as any function of the distance, to determine the apsidal angle.*

Assume $P = \mu u^2 \phi(u)$, then the differential equation of the orbit is

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2}\phi(u) = 0.$$

If the orbit were circular, we should have $u = c$, and $\frac{d^2u}{d\theta^2} = 0$, in which case $c - \frac{\mu}{h^2}\phi(c) = 0$........................$(a)$

When the orbit is very nearly circular we may put $u = c + x$, where x is always very small. Hence

$$\frac{d^2x}{d\theta^2} + c + x - \frac{\mu}{h^2}\phi(c+x) = 0,$$

$$\text{or } \frac{d^2x}{d\theta^2} + c + x - \frac{\mu}{h^2}\{\phi(c) + x\phi'(c)\} = 0, \text{ nearly;}$$

and (a) enables us to reduce this to

$$\frac{d^2x}{d\theta^2} + x\left(1 - \frac{\mu\phi'(c)}{h^2}\right) = 0,$$

or, by a second application of (a),

$$\frac{d^2x}{d\theta^2} + \left\{1 - \frac{c\phi'(c)}{\phi(c)}\right\}x = 0,$$

the integral of which is (§ 83)

$$x = A\cos\left[\sqrt{\left\{1 - \frac{c\phi'(c)}{\phi(c)}\right\}}\,\theta + B\right].$$

Hence the general value of θ which renders $\frac{dx}{d\theta} = 0$, is given by the equation

$$\sqrt{\left\{1 - \frac{c\phi'(c)}{\phi(c)}\right\}}\,\theta + B = n\pi,$$

n being any integer; and consequently the difference between any two successive values of θ is

$$\frac{\pi}{\sqrt{\left\{1 - \frac{c\phi'(c)}{\phi(c)}\right\}}},$$

the approximate apsidal angle.

Thus if the force vary directly as the n^{th} power of the distance, we have

$$\mu u^2\phi(u) = \mu u^{-n}; \text{ and } \phi(u) = u^{-n-2},$$

$$\text{whence } \phi'(u) = -(n+2)\,u^{-n-3},$$

$$\text{and the apsidal angle is } \frac{\pi}{\sqrt{(3+n)}}.$$

This shews that n cannot be less than -3, or that the force must not vary according to a higher inverse power of the distance than the third, if the circle is to be an approximation to the path of the particle: and furnishes a simple example of the determination of the conditions of *Kinetic Stability*, into which we cannot enter in this elementary treatise.

To find the law of force that the apsidal angle in the nearly circular orbit may be equal to a given angle, α suppose, we have

$$\frac{\pi}{\sqrt{\left\{1 - \frac{c\phi'(c)}{\phi(c)}\right\}}} = \alpha;$$

$$\text{from which } \frac{\phi'(c)}{\phi(c)} = \frac{1}{c}\left(1 - \frac{\pi^2}{\alpha^2}\right);$$

$$\text{or, by integration, } \log \frac{\phi(c)}{C} = \left(1 - \frac{\pi^2}{\alpha^2}\right) \log c,$$

$$\text{whence } \phi(c) = Cc^{1-\frac{\pi^2}{\alpha^2}};$$

and therefore the law of force, $\mu u^2 \phi(u)$, is $\mu u^{3-\frac{\pi^2}{\alpha^2}}$.

140. *A particle is projected from a given point in a given direction and with a given velocity, and moves under the action of a central force varying inversely as the square of the distance; to determine the orbit.*

We have $P = \mu u^2$, and therefore

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2} = 0,$$

$$\text{or } \frac{d^2}{d\theta^2}\left(u - \frac{\mu}{h^2}\right) + \left(u - \frac{\mu}{h^2}\right) = 0;$$

the integral of which is

$$u - \frac{\mu}{h^2} = A \cos(\theta + B),$$

or, as it is usually written,

$$u = \frac{\mu}{h^2}\{1 + e\cos(\theta - \alpha)\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$\text{This gives } \frac{du}{d\theta} = -\frac{\mu}{h^2} e \sin(\theta - \alpha) \quad \ldots\ldots\ldots\ldots\ldots (2).$$

Let R be the distance of the point of projection from the center; β the angle, and V the velocity, of projection; then when $\theta = 0$,

$$u = \frac{1}{R}, \quad \cot\beta = -\left(\frac{1}{u}\frac{du}{d\theta}\right)_{\theta=0} (\textit{Diff. Calc.});$$

$$\text{Hence, by (1), } \frac{h^2}{\mu R} - 1 = e\cos\alpha,$$

$$\text{and, by (2), } \frac{h^2}{\mu R}\cot\beta = -e\sin\alpha.$$

From these, $\tan\alpha = \dfrac{h^2 \cot\beta}{\mu R - h^2}$ .... (3),

and $e^2 = \dfrac{h^4}{\mu^2 R^2}\operatorname{cosec}^2\beta - \dfrac{2h^2}{\mu R} + 1$ .... (4).

But $h^2 = V^2 R^2 \sin^2\beta$, § 133;

wherefore $\tan\alpha = \dfrac{V^2 R \sin\beta\cos\beta}{\mu - V^2 R \sin^2\beta}$ .... (3′),

and $1 - e^2 = \dfrac{V^2 R^2 \sin^2\beta}{\mu}\left(\dfrac{2}{R} - \dfrac{V^2}{\mu}\right)$............ (4′).

Now (1) is the general polar equation of a conic section about the focus; and, as its nature depends on the value of the excentricity e given by (4′), we see that

if $V^2 > \dfrac{2\mu}{R}$, $e > 1$, and the orbit is a hyperbola,

$V^2 = \dfrac{2\mu}{R}$, $e = 1$, ... a parabola,

$V^2 < \dfrac{2\mu}{R}$, $e < 1$, ... an ellipse.

141. By § 96, the square of the velocity from infinity at distance R, for the law of force we are considering, is $\dfrac{2\mu}{R}$, and the above conditions may therefore be expressed more concisely by saying that the orbit will be a hyperbola, a parabola, or an ellipse, according as the velocity of projection is greater than, equal to, or less than, the velocity from infinity.

The velocity of a particle moving in a circle is also often taken as the standard of comparison for estimating the velocities of bodies in their orbits. For the gravitation law of force with which we are dealing the square of the velocity in a circle at distance R is $\dfrac{\mu}{R}$; and the above conditions may be expressed in another form by saying that the orbit will be

a hyperbola, a parabola, or an ellipse, according as the velocity of projection is greater than, equal to, or less than $\sqrt{2}$ times the velocity in a circle at the same distance.

142. Supposing the orbit to be an ellipse, we shall obtain its axis major and latus rectum most easily by a different process of integrating the differential equation. Multiplying it by $2h^2 \frac{du}{d\theta}$ and integrating, we obtain

$$h^2 \left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = v^2 = C + 2\mu u.$$

But when $u = \frac{1}{R}$, $v = V$; which gives $C = V^2 - \frac{2\mu}{R}$;

Hence $$h^2 \left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = v^2 = V^2 - \frac{2\mu}{R} + 2\mu u \quad \text{......} \quad (5).$$

Now to determine the apsidal distances, we must put

$$\frac{du}{d\theta} = 0;$$

and this gives us the condition

$$u^2 - \frac{2\mu}{h^2} u + \frac{2\mu}{h^2 R} - \frac{V^2}{h^2} = 0 \quad \text{..................} \quad (6),$$

which is a quadratic equation whose roots are the reciprocals of the two apsidal distances. But if a be the semiaxis major, and e the excentricity, these distances are

$$a(1-e) \text{ and } a(1+e).$$

Hence, as the coefficient of the second term of (6) is the sum of the roots with their signs changed, we have

$$\frac{1}{a(1-e)} + \frac{1}{a(1+e)} = \frac{2\mu}{h^2};$$

$$\text{or } a(1-e^2) = \frac{h^2}{\mu} \quad \text{........................} (7).$$

And, as the third term is the product of the roots,

$$\frac{1}{a^2(1-e^2)} = \frac{2\mu}{h^2R} - \frac{V^2}{h^2}:$$

$$\text{or } \frac{1}{a} = \frac{2}{R} - \frac{V^2}{\mu} \quad\quad\quad (8).$$

Substituting then $\frac{\mu}{a}$ for $\frac{2\mu}{R} - V^2$ in (5), we have

$$v^2 = \mu\left(2u - \frac{1}{a}\right) \quad\quad\quad (9).$$

Equations (7) and (8) give the latus rectum and axis major of the orbit, and shew that the axis major is independent of the direction of projection.

Equation (9) gives a useful expression for the velocity at any point.

143. The time of describing any given angle is to be obtained from the formula,

$$r^2\frac{d\theta}{dt} = h$$

$$= \sqrt{\{\mu a(1-e^2)\}}, \quad\quad \text{by equation (7).}$$

From this, combined with the polar equation to a conic section about the focus, we have

$$\frac{dt}{d\theta} = \frac{r^2}{\sqrt{\{\mu a(1-e^2)\}}}$$

$$= \sqrt{\frac{a^3(1-e^2)^3}{\mu}}\,\frac{1}{(1+e\cos\theta)^2};$$

measuring the angle from the nearest apse. To integrate this, let

$$\Theta = \frac{\sin\theta}{1+e\cos\theta}, \quad\quad \text{then}$$

$$\frac{d\Theta}{d\theta}=\frac{\cos\theta+e}{(1+e\cos\theta)^2}=\frac{\frac{1}{e}(1+e\cos\theta)-\frac{1-e^2}{e}}{(1+e\cos\theta)^2}$$

$$=\frac{1}{e}\frac{1}{1+e\cos\theta}-\frac{1-e^2}{e}\frac{1}{(1+e\cos\theta)^2};$$

$$\therefore \int\frac{d\theta}{(1+e\cos\theta)^2}=-\frac{e\Theta}{1-e^2}+\frac{1}{1-e^2}\int\frac{d\theta}{1+e\cos\theta}$$

$$=-\frac{e}{1-e^2}\frac{\sin\theta}{1+e\cos\theta}+\frac{1}{1-e^2}\int\frac{\sec^2\frac{\theta}{2}\,d\theta}{(1+e)+(1-e)\tan^2\frac{\theta}{2}};$$

$$=-\frac{e}{1-e^2}\frac{\sin\theta}{1+e\cos\theta}+\frac{2}{(1-e^2)^{\frac{3}{2}}}\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}\right\},$$

(if $e<1$);

$$\text{or}=\frac{e}{e^2-1}\frac{\sin\theta}{1+e\cos\theta}-\frac{1}{(e^2-1)^{\frac{3}{2}}}\log\left\{\frac{\sqrt{(e+1)}\cos\frac{\theta}{2}+\sqrt{(e-1)}\sin\frac{\theta}{2}}{\sqrt{(e+1)}\cos\frac{\theta}{2}-\sqrt{(e-1)}\sin\frac{\theta}{2}}\right\}$$

(if $e>1$).

Hence the time of describing, about the focus, an angle θ measured from the nearer apse is, in the ellipse,

$$\sqrt{\frac{a^3}{\mu}}\left[2\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}\right\}-e\sqrt{(1-e^2)}\frac{\sin\theta}{1+e\cos\theta}\right];$$

and, in the hyperbola,

$$\sqrt{\frac{a^3}{\mu}}\left[\log\left\{\frac{\sqrt{(e+1)}\cos\frac{\theta}{2}-\sqrt{(e-1)}\sin\frac{\theta}{2}}{\sqrt{(e+1)}\cos\frac{\theta}{2}+\sqrt{(e-1)}\sin\frac{\theta}{2}}\right\}+e\sqrt{(e^2-1)}\frac{\sin\theta}{1+e\cos\theta}\right].$$

144. In the parabola, if d be the apsidal distance, the integral becomes

$$\{\text{since } e=1,\ a(1-e)=d,\ a(1-e^2)=2d\},$$

$$t = \sqrt{\frac{8d^3}{\mu}} \int \frac{d\theta}{(1+\cos\theta)^2}$$

$$= \sqrt{\frac{8d^3}{\mu}} \int \frac{1}{4} \sec^4 \frac{\theta}{2} \, d\theta$$

$$= \sqrt{\frac{2d^3}{\mu}} \int \left(1 + \tan^2 \frac{\theta}{2}\right) d \tan \frac{\theta}{2}$$

$$= \sqrt{\frac{2d^3}{\mu}} \left(\tan \frac{\theta}{2} + \frac{1}{3} \tan^3 \frac{\theta}{2}\right).$$

145. From the result for the ellipse we see that the periodic time is $2\pi \sqrt{\frac{a^3}{\mu}}$. This might also have been found from the consideration that the periodic time is

$$T = \frac{2 \text{ area of ellipse}}{h} = \frac{2\pi a^2 \sqrt{(1-e^2)}}{\sqrt{\{\mu a (1-e^2)\}}}$$

$$= 2\pi \sqrt{\frac{a^3}{\mu}} = \frac{2\pi}{n},$$

in the notation commonly employed.

146. By laborious calculation from an immense series of observations of the planets, and of Mars in particular, Kepler enunciated the following, as the laws of the planetary motions about the Sun.

I. The planets describe Ellipses of which the Sun occupies a focus.

II. The radius vector of each planet traces out equal areas in equal times.

III. The squares of the periodic times of any two planets are as the cubes of the major axes of their orbits.

147. From the second of these laws we conclude that the planets are retained in their orbits by a central force tending to the Sun. For,

If the radius vector of a particle moving in a plane describe equal areas in equal times about a point in that plane, the resultant force on the particle tends to that point.

Take the point as origin, and let x, y be the co-ordinates of the particle at time t; X, Y the forces acting on it, resolved parallel to the axes; the equations of motion are

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y \text{..........................(1)}.$$

But by hypothesis, if A be the area traced out by the radius vector, $\frac{dA}{dt}$ is constant.

Hence, $$2\frac{dA}{dt} = x\frac{dy}{dt} - y\frac{dx}{dt} = C.$$

Differentiating, $$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0;$$

or, by (1), $$xY - yX = 0.$$

Hence, $\frac{Y}{X} = \frac{y}{x}$, and by the parallelogram of forces (§ 62) the resultant of X and Y passes through the origin.

148. From the first it follows that the law of the force is that of the inverse square of the distance.

The equation to an Ellipse about the focus is

$$u = \frac{2}{l}(1 + e\cos\theta),$$

where l is the latus rectum.

Hence, $\frac{d^2u}{d\theta^2} = -\frac{2e}{l}\cos\theta$, and therefore the force to the focus requisite for the description of the ellipse is (§ 127)

$$P = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$= \frac{2h^2}{l}u^2.$$

Hence, *if the orbit be an ellipse, described about a center of force at the focus, the law of force is that of the inverse square of the distance.*

149. From the third it follows that the force towards the Sun which acts on each of the planets is the same for each planet at the same distance.

For, in the formula in § 145, T^2 will not vary as a^3 unless μ be constant, i. e. unless the *absolute* force of the Sun be the same for all the planets.

We shall find afterwards (Chap. XII.) that for more reasons than one these laws are only approximate, but their enunciation was sufficient to enable Newton to propound the doctrine of Universal Gravitation; viz. that *every particle of matter in the universe attracts every other with a force which is as the masses directly, and as the square of the distance inversely.*

EXAMPLES.

(1) A particle describes an ellipse under the action of a force always directed to the center, to determine the law of the force.

From the polar equation to the ellipse, center pole

$$u^2 = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}; \text{ we have } u\frac{du}{d\theta} = \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\cos\theta\sin\theta;$$

$$\therefore u\frac{d^2u}{d\theta^2} + \left(\frac{du}{d\theta}\right)^2 = \left(\frac{1}{b^2} - \frac{1}{a^2}\right)(\cos^2\theta - \sin^2\theta);$$

$$\therefore P = \frac{h^2}{u}\left(u^4 + u^3\frac{d^2u}{d\theta^2}\right)$$

$$= \frac{h^2}{u}\left[u^4 - u^2\left(\frac{du}{d\theta}\right)^2 + u^2\left\{u\frac{d^2u}{d\theta^2} + \left(\frac{du}{d\theta}\right)^2\right\}\right]$$

$$= \frac{h^2}{u}\left\{u^4 - \left(\frac{1}{b^2} - \frac{1}{a^2}\right)^2\cos^2\theta\sin^2\theta + u^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\left(\cos^2\theta - \sin^2\theta\right)\right\}$$

$$= \frac{h^2}{u}\left\{u^2 + \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\cos^2\theta\right\}\left\{u^2 - \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\sin^2\theta\right\}$$

$$= \frac{h^2}{u}\cdot\frac{1}{b^2}\cdot\frac{1}{a^2} = \frac{h^2}{a^2b^2}r;$$

and therefore the law is that of the direct distance.

The above example is the converse of § 125. The solution may be very easily effected by the use of § 130.

(2) A particle describes a conic section under the action of a force always directed to one of the foci, to find the law of force.

In this case

$$u = \frac{1}{a(1-e^2)}\{1 + e\cos(\theta - \alpha)\},$$

$$\text{and } \therefore\ P = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$= \frac{h^2u^2}{a(1-e^2)} \propto \frac{1}{r^2},$$

the converse of § 140. See also § 148.

(3) Find the law of force, tending to the pole, under which a particle may describe an equiangular spiral.

$$P \propto \frac{1}{r^3}.$$

(4) Find the law of force by which a particle may describe the lemniscate of Bernoulli, the center of force being the node.

$$P \propto \frac{1}{r^7}.$$

(5) Find the law of force by which a particle may describe a circle, the center of force being in the circumference of the circle.

$$P \propto \frac{1}{r^5}.$$

(6) Find the law of force by which a particle may describe the spiral $r = a\left(\sec\frac{\theta}{n}\right)^n$, the centre of force being the pole of the spiral.

$$P \propto \frac{1}{r^{\frac{3n-2}{n}}}.$$

[Shew that this result is not true for $n = 1$, and find the correct one.]

(7) Find the law of force that a particle may describe the cissoid of Diocles $r = 2a\frac{\sin^2\theta}{\cos\theta}$, the center of force being the pole.

$$P \propto \frac{\operatorname{cosec}^2\theta}{r^3}.$$

[Here θ ought to be expressed in terms of r in the value of P.]

(8) A particle is projected from a given point in a given direction with the velocity which it would acquire in falling to the point of projection from an infinite distance, and is acted on by a force varying inversely as the n^{th} power of the distance, to determine the orbit.

Here $P = \mu u^n$, and therefore, § 127,

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2}u^{n-2} = 0.$$

Multiplying by $2h^2\frac{du}{d\theta}$ and integrating,

$$h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = v^2 = C + \frac{2\mu}{n-1}u^{n-1}.$$

Let V be the initial velocity, β the angle of projection, and c the initial value of u, then, § 96

$$V^2 = \frac{2\mu}{n-1}c^{n-1}.$$

But when $u=c$, $v=V$; $\therefore$ $C=0$, and

$$\left(\frac{du}{d\theta}\right)^2+u^2=\frac{2\mu u^{n-1}}{h^2(n-1)}.$$

Now § 133,

$$h^2=\frac{V^2\sin^2\beta}{c^2}=\frac{2\mu}{n-1}c^{n-3}\sin^2\beta;$$

$$\therefore \frac{\mu}{h^2}=\frac{n-1}{2c^{n-3}\sin^2\beta};$$

$$\therefore \left(\frac{du}{d\theta}\right)^2+u^2=\frac{u^{n-1}}{c^{n-3}\sin^2\beta};$$

$$\therefore \frac{d\theta}{du}=\frac{c^{\frac{n-3}{2}}\sin\beta}{u\sqrt{(u^{n-3}-c^{n-3}\sin^2\beta)}} \quad\text{.................}\quad (1).$$

To integrate this let

$$u^{n-3}=x^2;$$

$$\therefore (n-3)\,u^{n-4}\frac{du}{dx}=2x.$$

Dividing, $$\frac{n-3}{u}\frac{du}{dx}=\frac{2}{x}.$$

Substituting in (1),

$$\frac{d\theta}{dx}=\frac{2}{n-3}\frac{c^{\frac{n-3}{2}}\sin\beta}{x\sqrt{(x^2-c^{n-3}\sin^2\beta)}}.$$

Integrating,

$$\theta+C'=\frac{2}{n-3}\sec^{-1}\frac{x}{c^{\frac{n-3}{2}}\sin\beta}$$

$$=\frac{2}{n-3}\sec^{-1}\frac{u^{\frac{n-3}{2}}}{c^{\frac{n-3}{2}}\sin\beta}.$$

Suppose we take the initial line so that $C' = 0$, then

$$\frac{u^{\frac{n-3}{2}}}{c^{\frac{n-3}{2}} \sin\beta} = \sec\frac{n-3}{2}\theta;$$

or, if R be the initial distance $= \frac{1}{c}$,

$$\left(\frac{r}{R}\right)^{\frac{n-3}{2}} = \operatorname{cosec}\beta \cos\frac{n-3}{2}\theta,$$

the polar equation to the required path.

(9) A particle, acted on by a central force varying inversely as the fifth power of the distance, is projected in any direction with the velocity from infinity; find the orbit.

Its equation is $r = R \operatorname{cosec}\beta \sin(\beta - \theta)$, β being the angle of projection, and the line joining the point of projection with the center being taken as the initial line.

(10) A particle acted on by a central force varying inversely as the fifth power of the distance is projected from a given point with a velocity which is to the velocity from infinity as 5 to 3, in a direction making an angle $\sin^{-1}\frac{2\sqrt{6}}{5}$ with the radius vector; find the orbit.

Here we have

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2}u^3 = 0;$$

$$\therefore h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = v^2 = C + \frac{\mu u^4}{2}.$$

But if V be the velocity of projection, c the initial value of u,

$$V^2 = \frac{25}{9}\frac{\mu c^4}{2}; \text{ (§ 96)}:$$

$$\text{and when } u = c,\ v = V;\ \therefore\ C = \frac{8\mu c^4}{9};$$

$$\therefore \left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu}{h^2}\left(\frac{8c^4}{9} + \frac{u^4}{2}\right).$$

But $$h^2 = \frac{V^2 \sin^2 \beta}{c^2} = \frac{25\mu c^4}{18c^2}\frac{24}{25};$$

$$\therefore \frac{\mu}{h^2} = \frac{3}{4c^2}.$$

Substituting and integrating we find, after the necessary reductions,

$$r = \frac{\sqrt{3}}{2} R \frac{1 - \epsilon^{\sqrt{2}(\theta+\alpha)}}{1 + \epsilon^{\sqrt{2}(\theta+\alpha)}};$$

where R is the initial distance, and α a constant to be determined by the position of the initial line.

(11) A particle acted on by a force, varying partly as the inverse third, and partly as the inverse fifth, power of the distance, is projected with the velocity from infinity at an angle with the distance, the tangent of which is $\sqrt{2}$, the forces being equal at the point of projection; determine the orbit.

$$R - r = \frac{1}{\sqrt{2}} R\theta.$$

(12) The force tending to the center of a circle whose radius is a being $\mu\left(r + \frac{2a^3}{r^2}\right)$, find the velocity with which a particle will describe the circle; and shew that if the velocity be suddenly doubled the particle will come to an apse at the distance $3a$.

(13) If $P = 2\mu \frac{u^5}{c^2} + \mu u^3$, and a particle be projected at an angle of 45° with the initial distance $(R =) \frac{1}{c}$, with a velocity which is to the velocity in a circle at the same distance as $\sqrt{2}$ to $\sqrt{3}$, find the curve described.

$$r = R(1 - \theta).$$

(14) If a particle be acted on by a central force varying inversely as the seventh power of the distance, and be projected from an apse with a velocity which is to the velocity in a circle at the same distance as 1 to $\sqrt{3}$; find the equation to the curve described.

$$r^2 = R^2 \cos 2(\theta + \alpha).$$

(15) A particle, acted on by a force varying inversely as the cube of the distance, is projected from a given point with any velocity in any direction; to separate the curves according to the circumstances of projection. These curves are called *Cotes' Spirals.*

The equation of motion is

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2}u = 0 \text{ .... (1).}$$

Let $\frac{\mu}{h^2}$ be > 1, and let $\frac{\mu}{h^2} - 1 = k^2$; then

$$\frac{d^2u}{d\theta^2} - k^2u = 0,$$

the integral of which is

$$u = A\epsilon^{k\theta} + B\epsilon^{-k\theta} \text{ .... (2).}$$

This resolves itself into three distinct species of curves according to the values of A and B.

Species I. Let A and B have the same sign; then

$$u = A\epsilon^{k\theta} + B\epsilon^{-k\theta};$$

$$\text{and } \frac{du}{d\theta} = k(A\epsilon^{k\theta} - B\epsilon^{-k\theta}).$$

The values of A and B may in these equations be expressed in terms of the initial distance, and angle of projection; but we may put the equation to the curve in a simpler form as follows. Let α be the value of θ corresponding to an apse, then when $\theta = \alpha$, $\frac{du}{d\theta} = 0$;

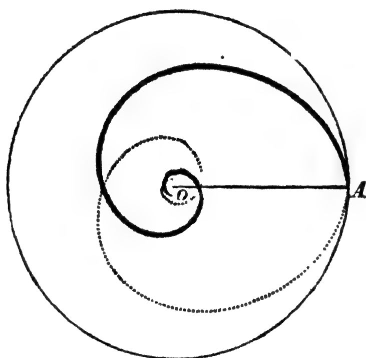
or $$0 = A\epsilon^{k\alpha} - B\epsilon^{-k\alpha},$$

which always gives a possible value of α; and therefore

$$A\epsilon^{k\alpha} = B\epsilon^{-k\alpha} \mp c, \text{ suppose.}$$

Substituting, $$u = c\left\{\epsilon^{k(\theta-\alpha)} + \epsilon^{-k(\theta-\alpha)}\right\}.$$

Hence when $\theta = \alpha$, $u = 2c$, or $\frac{1}{2c}$ is the apsidal distance. As θ increases, u increases, or r diminishes; and when $\theta = \infty$, $u = \infty$, or $r = 0$. Hence the curve forms an infinite number of convolutions about the pole; and, as it is symmetrical on both sides of the apse, it will be as represented in the figure, where A is the apse and O the center of force.



SPECIES II. Let $\frac{\mu}{h^2} > 1$, $B = 0$, then the equation (2) becomes

$$u = A\epsilon^{k\theta},$$

the equation of the logarithmic spiral. The nature of the curve will be the same if A, instead of B, vanish.

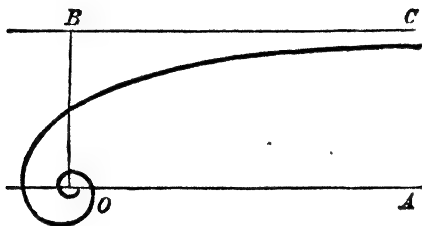
SPECIES III. Let $\frac{\mu}{h^2} > 1$, and B negative, then by equation (2),

$$u = A\epsilon^{k\theta} - B\epsilon^{-k\theta}.$$

Putting $u=0$, we obtain as for Species I,

$$u = c\left\{\epsilon^{k(\theta-\alpha)} - \epsilon^{-k(\theta-\alpha)}\right\}.$$

Hence, when $\theta=\alpha$, $u=0$ or $r=\infty$. As θ increases r decreases, and when θ is infinite $r=0$; so that there is an infinite number of convolutions round the pole. It is easily shewn that this curve has an asymptote parallel to OA, at a distance $\frac{1}{2ck}$.



SPECIES IV. Let $\frac{\mu}{h^2}=1$. Then equation (1) becomes

$$\frac{d^2u}{d\theta^2}=0,$$

the integral of which is

$$u=A(\theta-B),$$

the equation of the reciprocal spiral.

SPECIES V. Let $\frac{\mu}{h^2}<1$, and let $1-\frac{\mu}{h^2}=k^2$, then by equation (1),

$$\frac{d^2u}{d\theta^2}+k^2u=0,$$

the integral of which is

$$u=A\cos(k\theta+B);$$

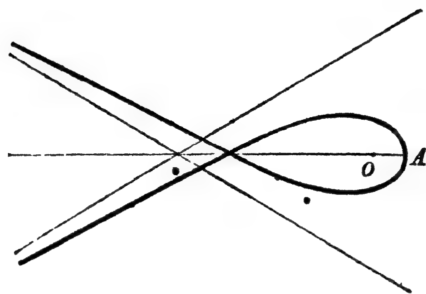
whence $$\frac{du}{d\theta}=-Ak\sin(k\theta+B).$$

Let α be the value of θ corresponding to the apse,

then $$k\alpha = -B;$$

and $$u = A \cos k(\theta - \alpha),$$

which shews that $\frac{1}{A}$ is the apsidal distance. The asymptotes to this curve are easily found for any assigned value of k. One case is represented in the annexed fig.



(16) A particle projected in any manner is acted on by a central force varying inversely as the fifth power of the distance, to determine the orbit.

Here $P = \mu u^5$, and we have

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2} u^3 = 0;$$

whence $$h^2 \left\{ \left(\frac{du}{d\theta}\right)^2 + u^2 \right\} = v^2 = C + \frac{\mu u^4}{2}.$$

When $u = c$, $v = V$; therefore $C = V^2 - \frac{\mu c^4}{2}$;

whence $$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{V^2}{h^2} - \frac{\mu c^4}{2h^2} + \frac{\mu u^4}{2h^2}.$$

But $h^2 = \frac{V^2 \sin^2 \beta}{c^2}$ (§ 133); and therefore

$$\left(\frac{du}{d\theta}\right)^2 = \frac{\mu u^4 c^2}{2V^2 \sin^2\beta} - u^2 + \frac{c^2}{\sin^2\beta} - \frac{\mu c^6}{2V^2 \sin^2\beta}.$$

This equation cannot be integrated in finite terms unless either the right-hand member be a perfect square, or $V^2 = \frac{\mu c^4}{2}$. The latter case is included in the more general one of Example (8). In the former we must have

$$\frac{2\mu c^2}{V^2 \sin^2\beta}\left(\frac{c^2}{\sin^2\beta} - \frac{\mu c^6}{2V^2 \sin^2\beta}\right) = 1,$$

$$\text{or } 2\mu\left(V^2 - \frac{\mu c^4}{2}\right) = \frac{V^4}{c^4}\sin^4\beta,$$

from which β may be found.

Extracting now the square root, integrating and taking the initial line so that the constant $=0$, we derive after the necessary reductions,

$$r = \frac{\sqrt{\mu}}{VR\sin\beta}\,\frac{1-\epsilon^{\theta\sqrt{2}}}{1+\epsilon^{\theta\sqrt{2}}}.$$

Giving $\sqrt{2}$ the positive or negative sign we have a spiral having an interior or an exterior asymptotic circle, the radius of this circle being in either case $\frac{\sqrt{\mu}}{VR\sin\beta}$.

(17) A particle is projected from an apse at the distance $\sqrt{(mh)}$, and is acted on by a central force $\frac{r}{m^2} + \frac{h^2}{r^3}$, h being twice the area described in a unit of time; find the equation to the orbit and the time of describing a given angle.

$$r^2 = \frac{mh}{1+\theta^2}, \quad t = m\tan^{-1}\theta.$$

(18) If a particle move about a center of force $\propto \frac{\mu}{r^2} + \frac{\nu}{r^3}$, shew that the equation to the orbit is generally of the form

$$r = \frac{a}{1 - e\cos(k\theta)}.$$

In the case when the projection takes place at an apse, the apsidal distance being $\frac{\mu}{h}$; and ν being equal to h^2, shew that the equation to the path is

$$r = \frac{2\mu h^2}{2h^3 + \mu^2\theta^2},$$

and that the time of describing an angle α is

$$\frac{1}{a}\tan\theta\,(\theta + \tfrac{1}{2}\sin 2\theta) \text{ where } \tan\theta = \frac{\mu\alpha}{\sqrt{(2h^3)}}.$$

(19) If v be the velocity, and P the force at distance r in a central orbit, and if v', P', r' be similar quantities for the corresponding point of the locus of the foot of the perpendicular on the tangent, shew that

$$\frac{v^2}{Pr} + \frac{P'r'}{v'^2} = 2.$$

(20) A particle attached to one end of an elastic string moves on a smooth horizontal plane, the other end of the string being fixed to a point in the plane. If the path of the particle be a circle, shew that the periodic time $\propto \left(\frac{ra}{r-a}\right)^{\frac{1}{2}}$, a and r being the natural and stretched lengths of the string. If the orbit be nearly circular, find the angle between the apsides.

(21) A particle is projected in such a manner as to describe a reciprocal spiral whose equation is $\theta = \frac{a}{r}$; shew that the time of performing the n^{th} revolution $= \frac{a^2}{2n(n-1)\pi\sqrt{\mu}}$.

(22) If P be a central force attracting a catenary, and p

be the perpendicular on the tangent at any point from the center of force; then the force which would cause a particle to revolve in the curve formed by the catenary $\propto \frac{P}{p}$.

(23) Find the time in which a particle would move from the vertex to the end of the latus rectum of a parabola; and shew that if the velocity be there suddenly altered in the ratio m to 1 (m being < 1) the body will proceed to describe an ellipse, the excentricity of which is $(1 - 2m^2 + 2m^4)^{\frac{1}{2}}$.

(24) A spherical surface is described in space, having in its center a force varying inversely as the square of the distance; shew that if a particle be let fall from this surface and be projected in any direction at any moment of its descent with the velocity acquired, it will move in an ellipse, the major axis of which is equal to the radius of the sphere.

(25) If the Earth's orbit be taken an exact circle, and a comet be supposed to describe round the Sun a parabolic orbit in the same plane; shew that the comet cannot possibly continue within the Earth's orbit longer than the $\left(\frac{2}{3\pi}\right)^{\text{th}}$ part of a year.

(26) If a particle be projected about a center of force varying inversely as the square of the distance, with a velocity equal to n times the velocity in a circle at the same distance; the angle α between the axis major and this distance may be determined from the equation

$$\tan(\alpha - \beta) = (1 - n^2)\tan\beta.$$

(27) A particle describes a parabola about a center of force ($\propto D^{-2}$) residing in a point in the circumference of a given ellipse the foci of which are in the circumference of the parabola; shew that the time of moving from one focus to the other is the same, at whatever point in the circumference of the ellipse the center of force is placed. (§ 165).

(28) A particle moves about a center of force, and its

velocity at any point is inversely proportional to the distance from the center of force; shew that its path will be a logarithmic spiral.

(29) A particle is describing a curve about a center of force, and its velocity $\propto \frac{1}{r^n}$, find the law of force and the equation to the path.

$$P \propto \frac{1}{r^{2n+1}}, \quad \left(\frac{r}{a}\right)^{n-1} = \cos\{(n-1)\theta + \alpha\}.$$

(30) A particle is projected in any direction from one extremity of a uniform straight line each particle of which attracts it with a force proportional to the distance, prove that the particle will pass through the other extremity.

(31) A particle projected in a given direction with a given velocity and attracted towards a given center of force has its velocity at every point to the velocity in a circle at the same distance as 1 to $\sqrt{2}$; find the orbit described, the position of the apse, and the law of force.

$$r = \sqrt{\frac{\mu}{2h^2}} \cos(\theta - \alpha), \quad P = \frac{\mu}{r^5}.$$

(32) A particle is projected from a given point with a given velocity and is acted on by a central force varying inversely as the square of the distance; shew that whatever be the direction of projection the center of the orbit described will lie on the surface of a certain sphere.

(33) Find the locus of the center of force that a cycloid may be described with uniform velocity, and find the law of force to the moving center.

(34) If a particle revolve in a circle of radius r, about a center of force distant a from the center of the circle, shew that the time from distance r to the nearer apse is

$$\frac{2^{\frac{3}{2}} r^{\frac{1}{2}}}{\sqrt{\phi}\left(2-\frac{a^2}{r^2}\right)^{\frac{3}{2}}}\left\{\cos^{-1}\frac{a}{2r}-\frac{a}{r}\sqrt{\left(1-\frac{a^2}{4r^2}\right)}\right\},$$

where ϕ is the initial force; and that the periodic time is

$$\frac{2\pi r^{\frac{3}{2}}}{(r-a)\sqrt{\phi}},$$

where ϕ is the force at the nearer apse.

(35) Shew that the only law of central force for which the velocity at each point of the orbit can be equal to that in a circle at the same distance is that of the inverse third power, and that the orbit is the logarithmic spiral.

(36) A particle describes an equilateral hyperbola about a center of force in the center, shew that an angle θ from the apsidal line is connected with the time t of its description by the formula

$$\sin 2\theta = \frac{\epsilon^{4\sqrt{\mu}t}-1}{\epsilon^{4\sqrt{\mu}t}+1}.$$

(37) If a number of particles, describing different circles in the same plane about a center of force $\propto D^{-3}$, start together from the same radius, find the curve in which they all lie when that which moves in the circle whose radius is a has completed a revolution.

(38) If the m^{th} power of the periodic time be proportional to the n^{th} power of the velocity in a circle, find the law of force in terms of the radius.

(39) If v be the velocity of a particle revolving in an ellipse about the center, v' its velocity when the direction of its motion is at right angles to the former direction, the time of describing the intercepted arc $= \frac{1}{\sqrt{\mu}}\sin^{-1}\frac{vv'}{\mu ab}$.

(40) A particle revolves in a circle about a center of force in the center, the force $\propto \frac{1}{D^2}$; the absolute force is suddenly increased in the ratio of $m : 1$ when the particle is at any assigned point of its path, and when the particle arrives again at the same point the absolute force is again increased in the same ratio; shew that the path which the particle will describe is an ellipse whose excentricity

$$= \frac{m^2 - 1}{m^2}.$$

(41) In a curve described by a particle under the action of a central force the angle between the radius vector and the tangent varies as the time. Find the curve and law of force.

(42) Shew that the apsidal angle is the same for different apsidal distances, only when the force is as some power of the distance.

(43) Given $P = \frac{2a^2h^2}{r^5} + \frac{h^2}{r^3}$, determine the path. Shew that in the particular case of the projection being made at distance a, and with velocity $= \frac{h}{a}\sqrt{2}$, the equation to the orbit is

$$r = a(1 + \theta).$$

(44) Force $= \frac{\mu}{r^4}$, and a particle is projected from an apse at distance a with velocity $= \sqrt{\frac{2\mu}{3a^3}}$, shew that the path is a cardioide, and that the periodic time is

$$\frac{3\pi}{4}\sqrt{\frac{3a^5}{2\mu}}.$$

(45) A particle is revolving in an ellipse about a center of force in the focus; supposing that every time the particle

arrives at the lower apse the absolute force is diminished in the ratio of 1 to $1-n$; find the excentricity of the elliptic orbit after p revolutions, the original excentricity being e.

$$\frac{1+e}{(1-n)^p}-1.$$

(46) A particle describes a circular orbit about a center of force situated in the center of the circle; prove that the form of the orbit will be stable or unstable according as the value of $\frac{d\log P}{d\log u}$, for $u=a$, is less or not less than 3, P being the central force, u the reciprocal of the radius vector, and $\frac{1}{a}$ the radius of the circle.

(47) If the equation for determining the apsidal distances in a central orbit contain the factor $(u-a)^p$, shew that $u=a$ cannot correspond to an apse unless p be of one of the forms $4m+2$ or $\frac{4m+2}{2n+1}$. If the factor $u-a$ occur twice, then a will be a root of the equation

$$\phi(u)-h^2u^3=0,$$

where $\phi(u)$ is the central force.

(48) Examine carefully the case of an apse where the center of force coincides with the center of curvature. Shew that the particle will, after passing such an apse, describe a circle about the center of force, but that the motion will be unstable.

CHAPTER VI.

ELLIPTIC MOTION.

150. In this chapter we propose to deduce from the results of the last some of the properties of Elliptic and Parabolic Orbits described about a center of force in the focus. This is a problem of great interest, as it has been proved by actual observation that the orbits of planets and comets are in general (neglecting the small effects of disturbing forces) ellipses either very slightly excentric, or so much so as to be scarcely distinguishable from parabolas. There are, it is true, some comets whose orbits are moderately excentric ellipses, and some whose orbits are hyperbolas; but, as the problem in their case becomes very complicated, and the approximate methods which we will here employ are inapplicable to their motions, it has been considered advisable to omit the consideration of such cases.

151. For the intelligibility of what follows it will be necessary to premise a few definitions.

Suppose APA' to be an elliptic orbit described about a center of force in the focus S. Also suppose P to be the position of the particle at any time t. Draw PM perpendicular to the major axis ACA', and produce it to cut the auxiliary circle in the point Q. Let C be the common center of the curves. Join CQ.

When the moving particle is at A, the nearest point of the orbit to S, it is said to be in *Perihelion*.

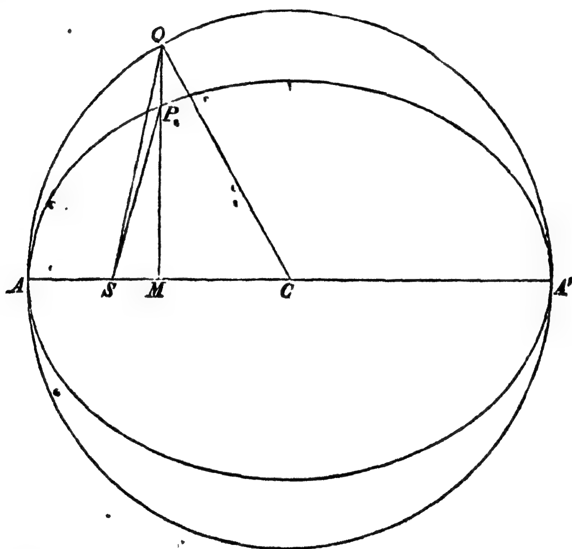
The angle ASP, or the excess of the particle's longitude over that of the perihelion, is called the *True Anomaly*. Let us denote it by θ.

The angle ACQ is called the *Excentric Anomaly*, and is generally denoted by u. And if $\frac{2\pi}{n}$ be the time of a complete

revolution, nt is the circular measure of an imaginary angle called the *Mean Anomaly;* it would evidently be the true anomaly if the particle's angular velocity about S were uniform.

152. It is easy from known properties of the ellipse to deduce relations between the mean and excentric, and also between the true and excentric, anomalies; and this we proceed to do.

To find the relation between the mean and excentric anomalies.



In the figure QCA is the excentric anomaly, and the mean anomaly is evidently to 2π as the area PSA is to the whole area of the elliptic orbit (§§ 145, 151), or as area QSA to area of auxiliary circle.

Now area QSA = area QCA − area QCS

$$= \tfrac{1}{2}a^2u - \tfrac{1}{2}a \, . \, ae \, . \sin u$$

(a being the semi-major axis of the orbit and e its excentricity)

$$=\frac{a^2}{2}(u-e\sin u).$$

$$\text{Hence } \frac{nt}{2\pi}=\frac{\frac{a^2}{2}(u-e\sin u)}{\pi a^2};$$

$$\text{or } nt=u-e\sin u.$$

153. *To find the relation between the true and excentric anomalies.*

We have (by Conics)

$$SP=\frac{a(1-e^2)}{1+e\cos\theta}.$$

$$\text{But } SP=a-eCM=a(1-e\cos u).$$

$$\text{Hence } \frac{1-e^2}{1+e\cos\theta}=1-e\cos u;$$

$$\text{Hence } \cos\theta=\frac{\cos u-e}{1-e\cos u},$$

$$\text{and } \tan\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$=\sqrt{\frac{1-e\cos u-\cos u+e}{1-e\cos u+\cos u-e}}$$

$$=\sqrt{\frac{(1+e)(1-\cos u)}{(1-e)(1+\cos u)}}$$

$$=\sqrt{\left(\frac{1+e}{1-e}\right)}\tan\frac{u}{2}.$$

The two equations which we have just found are sufficient for the solution of our problem, but they are sometimes obtained in the following manner.

154. The direct problem in elliptic motion is

To find the time of motion of a planet or comet through any portion of its elliptic orbit.

The equation of the orbit gives us

$$r = \frac{a(1-e^2)}{1+e\cos\theta}.$$

And from the description of equal areas in equal times, we have

$$\frac{d\theta}{dt} = \frac{h}{r^2} = \frac{\sqrt{\{\mu a(1-e^2)\}}}{r^2}.$$

From these equations we have

$$\frac{dt}{d\theta} = \frac{a^{\frac{3}{2}}(1-e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \frac{1}{(1+e\cos\theta)^2} = \frac{(1-e^2)^{\frac{3}{2}}}{n} \frac{1}{(1+e\cos\theta)^2}$$

if $\frac{2\pi}{n}$ be the period of revolution. (§ 145).

Hence if t_1 be the time of describing an arc measured by θ_1 from perihelion,

$nt_1 = (1-e^2)^{\frac{3}{2}} \int_0^{\theta_1} \frac{d\theta}{(1+e\cos\theta)^2}$, or (remarking that $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$, and $1 = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}$)

$$= (1-e^2)^{\frac{3}{2}} \int_0^{\theta_1} \frac{d\theta}{\left\{(1+e)\cos^2\frac{\theta}{2} + (1-e)\sin^2\frac{\theta}{2}\right\}^2}$$

$$= 2(1-e^2)^{\frac{3}{2}} \int_0^{\theta_1} \frac{\sec^2\frac{\theta}{2}\frac{d\tan\frac{\theta}{2}}{d\theta}\,d\theta}{\left\{(1+e)+(1-e)\tan^2\frac{\theta}{2}\right\}^2}.$$

To simplify this, let us put

$$\tan\frac{\theta}{2} = \sqrt{\left(\frac{1+e}{1-e}\right)}\tan\frac{\phi}{2} \text{ .... } (1),$$

(an assumption which will evidently conduct us to a formula already proved, as it is clear that ϕ will represent the excentric anomaly).

We have

$$nt_1 = 2(1-e^2)^{\frac{3}{2}}\int_0^{\phi_1} \frac{1+\frac{1+e}{1-e}\tan^2\frac{\phi}{2}}{(1+e)^2\sec^4\frac{\phi}{2}} \frac{d}{d\phi}\left\{\sqrt{\left(\frac{1+e}{1-e}\right)}\tan\frac{\phi}{2}\right\} d\phi$$

$$= \int_0^{\phi_1}\left\{(1-e)\cos^2\frac{\phi}{2} + (1+e)\sin^2\frac{\phi}{2}\right\} d\phi$$

$$= \int_0^{\phi_1}(1-e\cos\phi)\, d\phi.$$

$$= \phi_1 - e\sin\phi_1 \text{ .... } (2).$$

When θ_1 is given we must calculate ϕ_1 by means of (1), and thence t_1 by means of (2).

Since (1) and (2) give us the time of passing through an arc from perihelion subtending any angle θ_1 at the focus, it is evident that we have now the means of finding the time of describing any given portion of the orbit, and have thus the complete solution of the direct problem.

155. The inverse problem, which is far the most important, is to find the values of θ and r as functions of t, so that the direction and length of a planet's radius-vector may be determined for any given time. This generally goes by the name of Kepler's Problem.

156. Before entering on the systematic development of u, r and θ in terms of t from our equations, it may be useful to remark that if e be so small that higher terms than its

square may be neglected, we may easily obtain developments correct to the first three terms.

$$\text{Thus } u = nt + e \sin u$$

$$= nt + e \sin (nt + e \sin nt) \text{ nearly,}$$

$$= nt + e \sin nt + \frac{e^2}{2} \sin 2nt.$$

$$\text{Also } \frac{r}{a} = 1 - e \cos u$$

$$= 1 - e \cos (nt + e \sin nt)$$

$$= 1 - e \cos nt + \frac{e^2}{2} (1 - \cos 2nt).$$

$$\text{And } r^2 \frac{d\theta}{dt} = \sqrt{\{\mu a (1 - e^2)\}},$$

which may be written

$$\frac{a^2 (1 - e^2)^2}{(1 + e \cos \theta)^2} \frac{d\theta}{dt} = na^2 (1 - e^2)^{\frac{1}{2}},$$

$$\text{or } (1 - e^2)^{\frac{3}{2}} (1 + e \cos \theta)^{-2} \frac{d\theta}{dt} = n.$$

Keeping powers of e lower than the third

$$\left(1 - 2e \cos \theta + \frac{3}{2} e^2 \cos 2\theta\right) \frac{d\theta}{dt} = n,$$

$$\text{or } nt = \theta - 2e \sin \theta + \frac{3}{4} e^2 \sin 2\theta;$$

$$\text{whence } \theta = nt + 2e \sin \theta - \frac{3}{4} e^2 \sin 2\theta$$

$$= nt + 2e \sin (nt + 2e \sin nt) - \frac{3}{4} e^2 \sin 2nt$$

$$= nt + 2e \sin nt + 4e^2 \cos nt \sin nt - \frac{3}{4} e^2 \sin 2nt$$

$$= nt + 2e \sin nt + \frac{5}{4} e^2 \sin 2nt.$$

157. KEPLER'S PROBLEM. *To find r and θ as functions of t from the equations*

$$r = a(1 - e \cos u) \text{.....................} (1);$$

$$\tan \frac{\theta}{2} = \sqrt{\left(\frac{1+e}{1-e}\right)} \tan \frac{u}{2} \text{.................} (2);$$

$$nt = u - e \sin u \text{........................} (3).$$

These equations evidently give r, θ, and t directly for any assigned value of u, but this is of little value in practice. The method of solution which we proceed to give is that of Lagrange, and the general principle of it is this—

We can develop θ from equation (2) in a series ascending by powers of a small function of e, the coefficients of these powers being u and the sines of multiples of u. Now by Lagrange's theorem we may from equation (3) express u, $1 - e \cos u$, $\sin u$, $\sin 2u$, &c. in series ascending by powers of e, and whose coefficients are sines or cosines of multiples of nt. Hence by substituting these values in equation (1) and in the development of (2), we have r and θ expressed in series whose terms rapidly decrease, and whose coefficients are sines or cosines of multiples of nt. And this is the complete practical solution of the problem.

158. *To express the true, as a function of the excentric, anomaly.*

Substituting in (2) the exponential expressions for the tangents, and writing i for $\sqrt{-1}$, we have

$$\frac{\epsilon^{\frac{i\theta}{2}} - \epsilon^{-\frac{i\theta}{2}}}{\epsilon^{\frac{i\theta}{2}} + \epsilon^{-\frac{i\theta}{2}}} = \sqrt{\left(\frac{1+e}{1-e}\right)} \frac{\epsilon^{\frac{iu}{2}} - \epsilon^{-\frac{iu}{2}}}{\epsilon^{\frac{iu}{2}} + \epsilon^{-\frac{iu}{2}}},$$

whence

$$\epsilon^{i\theta} = \frac{\epsilon^{iu}\{\sqrt{(1+e)}+\sqrt{(1-e)}\}+\{\sqrt{(1-e)}-\sqrt{(1+e)}\}}{\epsilon^{iu}\{\sqrt{(1-e)}-\sqrt{(1+e)}\}+\{\sqrt{(1-e)}+\sqrt{(1+e)}\}};$$

or, putting $\lambda = \dfrac{\sqrt{(1+e)}-\sqrt{(1-e)}}{\sqrt{(1+e)}+\sqrt{(1-e)}} = \dfrac{e}{1+\sqrt{(1-e^2)}}$,

$$\epsilon^{i\theta} = \epsilon^{iu} \cdot \frac{1-\lambda\epsilon^{-iu}}{1-\lambda\epsilon^{iu}}.$$

Taking the logarithm of each side and dividing by i,

$$\theta = u + \frac{\lambda}{i}\{\epsilon^{iu} - \epsilon^{-iu}\} + \frac{\lambda^2}{2i}\{\epsilon^{2iu} - \epsilon^{-2iu}\} + \dots$$

$$= u + 2\left(\lambda \sin u + \frac{\lambda^2}{2}\sin 2u + \frac{\lambda^3}{3}\sin 3u + \&c.\right) \dots\dots (4).$$

159. *To develop u in terms of t.*

If we have

$$y = z + x\phi(y) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (5),$$

we obtain, by Lagrange's Theorem, the development

$$f(y) = f(z) + x\phi(z)f'(z) + \frac{x^2}{1.2}\frac{d}{dz}\{\overline{\phi(z)}|^2 f'(z)\}$$

$$+ \frac{x^3}{1.2.3}\left(\frac{d}{dz}\right)^2\{\overline{\phi(z)}|^3 f'(z)\} + \&c. \dots\dots\dots (6).$$

Now equation (3) may be put in the form

$$u = nt + e\sin u,$$

which is identical with (5) if

$$y = u,\ z = nt,\ x = e,\ \text{and}\ \phi(y) = \sin y.$$

Also, as it is the development of u that we require, we must put

$$f(u) = u, \text{ and } f'(u) = 1. \quad \text{Hence, by (6)}$$

$$y = z + x \sin z + \frac{x^2}{1.2} \frac{d}{dz} (\sin^2 z) + \frac{x^3}{1.2.3} \left(\frac{d}{dz}\right)^2 (\sin^3 z) + \&c.;$$

and, substituting for the powers of $\sin z$ their corresponding expressions in sines and cosines of multiples of z,

$$y = z + x \sin z + \frac{x^2}{1.2} \frac{d}{dz} \left(\frac{1 - \cos 2z}{2}\right) + \frac{x^3}{1.2.3} \left(\frac{d}{dz}\right)^2 \left(\frac{3 \sin z - \sin 3z}{4}\right)$$

$$+ \frac{x^4}{1.2.3.4} \left(\frac{d}{dz}\right)^3 \left(\frac{3 - 4 \cos 2z + \cos 4z}{8}\right) + \&c.$$

$$= z + x \sin z + \frac{x^2}{2} \sin 2z + \frac{x^3}{8} (3 \sin 3z - \sin z) + \ldots\ldots$$

or, substituting for x, y, z their values as above,

$$u = nt + e \sin nt + \frac{e^2}{2} \sin 2nt + \frac{e^3}{8} (3 \sin 3nt - \sin nt)$$

$$+ \frac{e^4}{6} (2 \sin 4nt - \sin 2nt) + \&c. \ldots\ldots (7).$$

To develop $\sin u$, we recur to equation (3), which gives, after the elimination of u by means of (7),

$$\sin u = \sin nt + \frac{e}{2} \sin 2nt + \frac{e^2}{8} (3 \sin 3nt - \sin nt) + \&c. \ldots. (8).$$

By the application of Lagrange's theorem to equation (3), it is easy to deduce the following expressions:

$$\sin 2u = \sin 2nt + e (\sin 3nt - \sin nt) + e^2 (\sin 4nt - \sin 2nt)$$

$$+ \frac{e^3}{24} (4 \sin nt - 27 \sin 3nt + 25 \sin 5nt) + \&c.$$

$$\sin 3u = \sin 3nt + \frac{3e}{2}(\sin 4nt - \sin 2nt)$$

$$+ \frac{e^2}{8}(15 \sin 5nt - 18 \sin 3nt + 3 \sin nt) + \&c.$$

&c. = &c.

Substituting these values in (4), we obtain the value of θ, containing however the quantity λ. If we take as its approximate value $\frac{e}{2} + \frac{e^3}{8}$, and make the requisite substitutions, we obtain

$$\theta = nt + (2e - \frac{1}{4}e^3) \sin nt + \frac{5}{4}e^2 \sin 2nt + \frac{13}{12}e^3 \sin 3nt + \ldots\ldots$$

which is correct as far as e^3.

160. In proceeding farther with the development, it becomes necessary to expand λ and its powers in series ascending by powers of e. This is readily done as follows.

We have

$$\lambda = \frac{e}{1 + \sqrt{(1 - e^2)}} = \frac{e}{E} \text{ suppose.}$$

Hence $E = 2 - \frac{e^2}{E}$, from which by Lagrange's Theorem,

$$E^{-p} = \frac{1}{2^p} + \frac{p}{2^{p+2}}e^2 + \frac{p \cdot (p+3)}{2 \cdot 2^{p+4}}e^4 + \&c.;$$

and thus the value of λ^p, being $e^p E^{-p}$, is known.

The correct value of θ to the fifth power of e is thus found to be

$$nt + 2e\sin nt + \frac{5e^2}{4}\sin 2nt + \frac{e^3}{2^2.3}(13\sin 3nt - 3\sin nt)$$

$$+\frac{e^4}{2^5.3}(103\sin 4nt - 44\sin 2nt)$$

$$+\frac{e^5}{2^6.3.5}(1097\sin 5nt - 645\sin 3nt + 50\sin nt).$$

161. *To develop r in terms of t.*

From (1) it is evident that all we have to do is to develop, by Lagrange's Theorem, $1 - e\cos u$ as a function of t, from $nt = u - e\sin u$.

To develop $(1 - e\cos u)$ in terms of t.

Here $f(y) = 1 - e\cos y$,

$f'(y) = e\sin y$;

and the form of ϕ is the same as before; hence

$$1 - e\cos y = (1 - e\cos z) + x\sin z\,(e\sin z)$$

$$+\frac{x^2}{1.2}\frac{d}{dz}(\sin^2 z \,.\, e\sin z) + \ldots\ldots\ldots$$

Hence, as before, substituting for the powers of sines their equivalent expressions in sines and cosines of multiple arcs, differentiating, and substituting u for y, nt for z, and e for x, we have

$$1 - e\cos u = \frac{r}{a} = 1 - e\cos nt + \frac{e^2}{2}(1 - \cos 2nt)$$

$$+\frac{e^3}{8}(3\cos nt - 3\cos 3nt)$$

$$+\frac{e^4}{3}(\cos 2nt - \cos 4nt) + \&c.$$

which gives the radius vector in terms of the time.

162. In the case of parabolic motion the above methods are not applicable; but a much simpler one is.

To find the time of describing any arc of a parabola from the vertex; the center of force being in the focus.

The equation to the curve is

$r = d\sec^2\dfrac{\theta}{2}$, where d is the perihelion distance.

And the condition of equable description of areas gives

$$r^2\frac{d\theta}{dt} = h = \sqrt{(2\mu d)};$$

$$t_1 = \frac{d^{\frac{3}{2}}}{\sqrt{(2\mu)}}\int_0^{\theta_1}\sec^4\frac{\theta}{2}\,d\theta$$

$$= \frac{\sqrt{2}d^{\frac{3}{2}}}{\sqrt{\mu}}\int_0^{\theta_1}\left(1+\tan^2\frac{\theta}{2}\right)\frac{d\tan\frac{\theta}{2}}{d\theta}\,d\theta$$

$$= \frac{\sqrt{2}d^{\frac{3}{2}}}{\sqrt{\mu}}\left(\tan\frac{\theta_1}{2}+\frac{1}{3}\tan^3\frac{\theta_1}{2}\right);$$

$$\text{or } nt_1 = \left(\tan\frac{\theta_1}{2}+\frac{1}{3}\tan^3\frac{\theta_1}{2}\right), \text{ if } \sqrt{\frac{\mu}{2d^3}} = n;$$

which is the expression required. From this it is evidently easy to calculate the time of describing any arc of the orbit.

The inverse problem of parabolic motion would require the solution of the cubic equation just found for $\tan\dfrac{\theta}{2}$ in terms of the time. This however is easily obviated by the formation of a table in which corresponding values of t and $\dfrac{\theta}{2}$ are calculated on the supposition that $n = 1$. If we wish then to solve the inverse problem, all we have to do is to find the

value of θ corresponding to the number nt. This will be the true anomaly required, and the same table will of course apply to any parabolic orbit.

163. If the orbit be not parabolic, but elliptic and of very great excentricity, the result of the following direct problem is sometimes of use.

To find the place of a comet at a given time in a very excentric elliptic orbit.

We have $\dfrac{dt}{d\theta} = \dfrac{a^{\frac{3}{2}}(1-e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \dfrac{1}{(1+e\cos\theta)^2}$. (§ 154).

Let D be the perihelion distance, $D = a(1-e)$;

$$\frac{dt}{d\theta} = \frac{D^{\frac{3}{2}}(1+e)^{\frac{3}{2}}}{\sqrt{\mu}} \cdot \frac{\sec^4\dfrac{\theta}{2}}{\left\{(1+e)+(1-e)\tan^2\dfrac{\theta}{2}\right\}^2}$$

$$= \frac{D^{\frac{3}{2}}}{\sqrt{\{\mu(1+e)\}}} \sec^4\frac{\theta}{2}\left\{1+\frac{1-e}{1+e}\tan^2\frac{\theta}{2}\right\}^{-2}.$$

Expanding in powers of $(1-e)$, and neglecting higher powers of $(1-e)$ than the first; since $e \doteq 1$ nearly,

$$nt_1 = \frac{1}{2}\left(1-\frac{1-e}{2}\right)^{-\frac{1}{2}} \int_0^{\theta_1} \sec^4\frac{\theta}{2}\left\{1-(1-e)\tan^2\frac{\theta}{2}\right\} d\theta$$

$$= \int_0^{\theta_1} \frac{d\tan\dfrac{\theta}{2}}{d\theta} \left\{1+\tan^2\frac{\theta}{2}+(1-e)\left(\frac{1}{4}-\frac{3}{4}\tan^2\frac{\theta}{2}-\tan^4\frac{\theta}{2}\right)\right\} d\theta;$$

whence

$$nt_1 = \tan\frac{\theta_1}{2}+\frac{1}{3}\tan^3\frac{\theta_1}{2}+(1-e)\left(\frac{1}{4}\tan\frac{\theta_1}{2}-\frac{1}{4}\tan^3\frac{\theta_1}{2}-\frac{1}{5}\tan^5\frac{\theta_1}{2}\right) \ldots (1).$$

The following is a convenient method of calculating the value of θ_1 for a given value of t_1.

Suppose $\bar{\theta}$ to be at time t_1, the true anomaly of a comet moving in a parabolic orbit of which D is the perihelion distance; then by § 162

$$nt_1 = \tan\frac{\bar\theta}{2} + \frac{1}{3}\tan\frac{\bar\theta}{2} \dots\dots\dots\dots\dots (2).$$

Let $\theta_1 = \bar\theta + x$, substitute in equation (1) and (since x is very small) expand in powers of x by Taylor's Theorem: we have approximately

$$nt_1 = \tan\frac{\bar\theta}{2} + \frac{1}{3}\tan^3\frac{\bar\theta}{2} + \frac{1}{2}x\sec^4\frac{\bar\theta}{2} + \dots\dots$$

$$+\frac{1}{4}(1-e)\tan\frac{\bar\theta}{2}\left(1 - \tan^2\frac{\theta}{2} - \frac{4}{5}\tan^4\frac{\bar\theta}{2}\right).$$

From this, by means of (2), we obtain

$$x = \frac{1}{10}(1-e)\tan\frac{\bar\theta}{2}\left(4 - 3\cos^2\frac{\bar\theta}{2} - 6\cos^4\frac{\bar\theta}{2}\right).$$

To make use of this formula, there must be added to the table before mentioned, a column giving the values of $\frac{x}{1-e}$ corresponding to those of $\bar\theta$. Taking then any value of t, we seek in the second column the value of $\bar\theta$ for the number nt, and then the value of $\frac{x}{1-e}$ for the value of $\bar\theta$ so found.

As the orbit is known, $1-e$ is known, hence x and $\bar\theta$ are known in terms of t, and the true anomaly, or

$\bar\theta + x$, is known.

164. *Remark.* In all that precedes we have supposed for simplicity that the angle θ, which determines the position of the particle, is measured from perihelion; and that $\theta = 0$, $t = 0$, together. This is not usually the case, but let θ' be the *longitude* of the particle at time t, ϖ that of the perihelion, ϵ the longitude of the particle at time $t = 0$, or the *Epoch* as it is generally called; then at time t the mean longitude is

evidently $nt + \epsilon$, and the mean anomaly $nt + \epsilon - \varpi$. Hence by our previous results

$$\theta' - \varpi = nt + \epsilon - \varpi + 2e \sin(nt + \epsilon - \varpi)$$

$$+ \frac{5}{4} e^2 \sin 2(nt + \epsilon - \varpi) + \&c.$$

$$r = a\{1 - e\cos(nt + \epsilon - \varpi) + \ldots\},$$

which are the formulæ in general use; θ' being, as before observed, the true longitude at time t.

165. *The time through any arc of a parabolic orbit, described about the focus, may be expressed in terms of the chord and extreme radii vectores of the arc.*

Let r_1, θ_1, r_2, θ_2 be the co-ordinates of the extreme points, c the chord, of the arc. Then, T being the required time, we have (§ 162)

$$nT = \tan\frac{\theta_2}{2} - \tan\frac{\theta_1}{2} + \frac{1}{3}\left(\tan^3\frac{\theta_2}{2} - \tan^3\frac{\theta_1}{2}\right),$$

or, as we may write it for simplicity,

$$= t_2 - t_1 + \frac{1}{3}(t_2^3 - t_1^3)$$

$$= \frac{1}{3}(t_2 - t_1)(3 + t_1^2 + t_2^2 + t_1 t_2)$$

$$= \frac{1}{3}(t_2 - t_1)\{(1 + t_1^2) + (1 + t_2^2) + (1 + t_1 t_2)\}.$$

$$\text{Now } 1 + t_1 t_2 = \frac{\cos\frac{\theta_2 - \theta_1}{2}}{\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}},$$

and in the triangle whose base is c, sides r_1, r_2, and vertical angle $\theta_2 - \theta_1$, we have by Trigonometry

$$\cos\frac{\theta_2-\theta_1}{2}=\sqrt{\frac{s(s-c)}{r_1r_2}},\quad\left(\text{where } s=\frac{r_1+r_2+c}{2}\right).$$

$$\text{Also,}\quad r_1=d\sec^2\frac{\theta_1}{2},\quad r_2=d\sec^2\frac{\theta_2}{2}.$$

$$\text{Hence,}\quad 1+t_1t_2=\frac{1}{d}\sqrt{\{s(s-c)\}}.$$

And $$t_2-t_1=\sqrt{\{1+t_2^2+1+t_1^2-2(1+t_1t_2)\}}$$

$$=\sqrt{\frac{1}{d}[r_2+r_1-2\sqrt{\{s(s-c)\}}]}$$

$$=\sqrt{\frac{1}{d}[2s-c-2\sqrt{\{s(s-c)\}}]}$$

$$=\frac{1}{\sqrt{d}}\{\sqrt{s}-\sqrt{(s-c)}\}.$$

Also $$1+t_1^2+1+t_2^2+1+t_1t_2=\frac{1}{d}[r_1+r_2+\sqrt{\{s(s-c)\}}]$$

$$=\frac{1}{d}[2s-c+\sqrt{\{s(s-c)\}}].$$

Hence $$nT=\frac{1}{3d^{\frac{3}{2}}}\{\sqrt{s}-\sqrt{(s-c)}\}\{s+\sqrt{s}\sqrt{(s-c)}+(s-c)\}$$

$$=\frac{1}{3d^{\frac{3}{2}}}\{s^{\frac{3}{2}}-(s-c)^{\frac{3}{2}}\}.$$

$$\text{But}\quad n=\sqrt{\frac{\mu}{2d^3}},\ \text{by § 162.}$$

Substituting this, and the values of s, $(s-c)$, we have

$$T=\frac{1}{6\sqrt{\mu}}\{(r_1+r_2+c)^{\frac{3}{2}}-(r_1+r_2-c)^{\frac{3}{2}}\}.$$

In this investigation we have supposed the arc not to include the perihelion; if it should do so we must take the *sum* of the radicals as the value of T.

166. It may be shewn in a similar manner that the time of describing about the focus an arc of an ellipse or hyperbola whose chord and extreme radii vectores are given, may be expressed in terms of these quantities and the axis major alone. For the proof we must refer to the *Mécanique Céleste*, or to Pontécoulant's *Système du Monde*.

It may also be shewn, in much the same manner, that the ratio of the area described in a given time, to that of the triangle formed by the chord and extreme radii vectores, may be expressed independently of the parameter of the path.

EXAMPLES.

(1) If the perihelion distance of a comet's orbit be $\frac{1}{3}$ of the radius of the Earth's orbit supposed circular, find the number of days the comet will remain within the Earth's orbit.

(2) If a comet describe 90° from perihelion in 100 days, compare its perihelion distance with the distance of a planet which describes its circular orbit in 942 days.

(3) Shew how to divide a planet's elliptic orbit by a diameter, so that the times of describing the two parts are as $n : 1$, and find in what cases only one such line can be drawn.

(4) In the case of planets and comets prove the following formulæ, the letters being the same as in the text,

$$r\frac{d\theta}{du} = a\sqrt{(1-e^2)};$$

$$\frac{r}{a}\sin\theta = \frac{\sqrt{(1-e^2)}}{e}(u-nt);$$

$$\log\frac{r}{a} = -\log(1+\lambda^2)$$
$$-2\left(\lambda\cos u + \tfrac{1}{2}\lambda^2\cos 2u + \tfrac{1}{3}\lambda^3\cos 3u + \&c.\right)$$

(5) A body describes an ellipse: prove that the times of describing the two parts, into which the orbit is divided by the axis minor, are to one another as $\pi + 2e$ is to $\pi - 2e$, where e is the excentricity of the ellipse.

(6) If Pp, Qq be chords parallel to the axis major of an elliptic orbit, shew that the difference of the times through the arcs PQ, pq varies as the distance between the chords.

(7) If a comet whose orbit is inclined to the plane of the ecliptic were observed to pass over the Sun's disc, and three months after to strike the planet Mars, determine its distance from the Earth at the first observation, the Earth and Mars describing about the Sun circles in the same plane whose radii are as 2 : 3.

(8) Shew that the arithmetic mean of the distances of a planet from the Sun, at equal indefinitely small intervals of time, is

$$a\left(1 + \frac{e^2}{2}\right).$$

(9) When a body describes an ellipse under the action of a force in the focus S, if H be the other focus, the square of the velocity at P varies as $\frac{HP}{SP}$.

(10) The time through an arc of a parabolic orbit bounded by a focal chord $\propto$ (chord)$^{\frac{3}{2}}$.

(11) If a circle be described passing through the focus and vertex of a parabolic orbit, and also through the position of the moving particle at each instant, shew that its center describes with uniform velocity a straight line bisecting at right angles the perihelion distance.

(12) Shew that the velocity of a comet perpendicular to the major axis varies inversely as its radius vector.

(13) D_1, D_2 being two distances of a comet, on opposite sides of perihelion, including a known angle, shew that the position of perihelion may be found from the equation.

$$\frac{\sqrt{D_1} - \sqrt{D_2}}{\sqrt{D_1} + \sqrt{D_2}} = \tan\tfrac{1}{4}\,(\text{sum of true anomalies})\,.\tan\tfrac{1}{4}\,(\text{difference}).$$

(14) In what point of all conic sections is the paracentric velocity a maximum? Shew that in such a case the velocity is to that in a circle at the same distance as the distance is to the perpendicular on the tangent.

(15) In an elliptic orbit find the relation between the mean angular velocity about the center of force and the angular velocity about the other focus, and thence shew that when e is small the latter is nearly uniform.

(16) If α, β be the greatest and least angular velocities in an ellipse about the focus, the *mean* angular velocity is

$$\frac{2\sqrt[4]{(\alpha^3\beta^3)}}{\sqrt{\alpha}+\sqrt{\beta}}.$$

(17) Find the maximum value of $\theta - nt$ in an elliptic orbit, and develop it in powers of e, shewing that it cannot contain even powers.

If Θ be this quantity,

$$\Theta = 2e + \frac{11e^3}{3 \cdot 2^4} + \frac{599e^5}{5 \cdot 2^{10}} + \&c.$$

CHAPTER VII.

CONSTRAINED MOTION.

167. WE come now to the case of the motion of a particle subject to the action not only of given forces, but of undetermined pressures or tensions. Such cases occur when the particle is attached to a fixed, or moving, point by means of a rod or string, and when it is forced to move on a curve or surface.

In applying to a problem of this kind the general equations of motion of a free particle, we must assume directions and intensities for the unknown forces, treating them then as known, and it will always be found that the geometrical circumstances of the motion will furnish the requisite number of additional equations for the determination of all the unknown quantities in terms of the time.

One case of this kind has been already treated of (§ 79), namely, that of a particle moving on an inclined plane under the action of gravity. There the undetermined force is the pressure on the plane, which however is evidently constant, and equal to the resolved part of the particle's weight perpendicular to the plane.

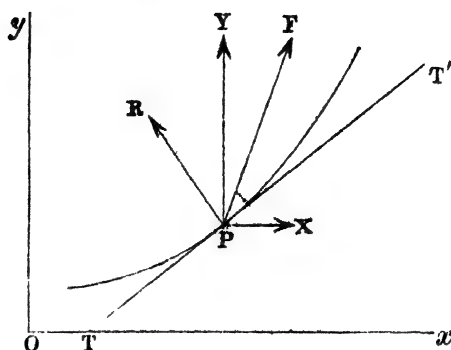
The laws of kinetic friction are but imperfectly known, and the few investigations which will be given of motion on a rough curve or surface are of very slight importance.

168. The simplest case is

A particle is constrained to move on a given smooth plane curve, under the action of given forces in the plane of the curve, to determine the motion.

Taking rectangular axes in this plane, the forces may be resolved into two, X, Y, parallel respectively to the axes of x and y. In addition there will be the force R, the mutual pressure between the curve and particle, which evidently acts in the normal to the curve since there is no friction.

Let P be the position of the particle at the time t; and let



the forces X, Y, R, act as in the figure. Draw PT, a tangent to the constraining curve at P. Then if $PTx = \theta$, we have

$$\sin\theta = \frac{dy}{ds}, \quad \cos\theta = \frac{dx}{ds}.$$

The mass of the particle being, as before, taken as unity, the equations of motion are

$$\frac{d^2x}{dt^2} = X - R\sin\theta = X - R\frac{dy}{ds} \quad\text{..............} (1),$$

$$\frac{d^2y}{dt^2} = Y + R\cos\theta = Y + R\frac{dx}{ds} \quad\text{..............} (2).$$

These two equations, together with the equation to the given curve, are sufficient to determine the motion completely.

To eliminate R, multiply (1) by $\frac{dx}{dt}$, (2) by $\frac{dy}{dt}$, and add. We thus obtain,

$$\left(\text{since } \frac{dy}{ds}\frac{dx}{dt} = \frac{dx}{ds}\frac{dy}{dt}\right),$$

$$\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} = \frac{ds}{dt}\frac{d^2s}{dt^2} = X\frac{dx}{dt} + Y\frac{dy}{dt} \quad\text{........} (3),$$

or, as we may write it,

$$\frac{d^2s}{dt^2} = X\frac{dx}{ds} + Y\frac{dy}{ds},$$

which might at once have been obtained by expressing the acceleration along the tangent.

Now, it has been shewn in Chap. II. that if the forces resolved into X and Y are such as occur in nature,

$$Xdx + Ydy$$

is the complete differential of some function $\phi(x, y)$. See § 73.

Integrating (3) on this hypothesis, we have

$$\frac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} = \frac{v^2}{2} = \phi(x, y) + C \text{..............(4)},$$

supposing v to represent the entire velocity of the particle at time t.

Imagine the particle to start at the time $t = 0$, from a point whose co-ordinates are a, b, with a velocity V.

We have, from (4),

$$\frac{1}{2}V^2 = \phi(a, b) + C;$$

and therefore $\frac{1}{2}v^2 = \frac{1}{2}V^2 + \phi(x, y) - \phi(a, b)$ ....(5).

This shews that a particle, constrained to move under the action of the forces X, Y, along any path whatever from the point a, b to the point x, y, has on arriving at the latter point, the square of its velocity increased by a quantity entirely independent of the path pursued: another simple case of the conservation of energy.

169. *To find the pressure on the curve.*

Multiply equation (1) by $\frac{dy}{dt}$, (2) by $\frac{dx}{dt}$, and subtract.

Then, noticing that

$$\frac{dy}{ds}\frac{dy}{dt} + \frac{dx}{ds}\frac{dx}{dt} = \frac{\left(\frac{ds}{dt}\right)^2}{\frac{ds}{dt}} = \frac{ds}{dt},$$

we have

$$\frac{dy}{dt}\frac{d^2x}{dt^2}-\frac{dx}{dt}\frac{d^2y}{dt^2}=X\frac{dy}{dt}-Y\frac{dx}{dt}-R\frac{ds}{dt}.$$

But (*Diff. Calc.*) if ρ be the radius of curvature of the constraining curve at the point x, y,

$$\rho=\frac{\left(\frac{ds}{dt}\right)^2}{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}.$$

Transforming by means of this, the above equation becomes

$$-\frac{\left(\frac{ds}{dt}\right)^2}{\rho}=X\frac{dy}{ds}-Y\frac{dx}{ds}-R;$$

$$\text{or, } R=X\sin\theta-Y\cos\theta+\frac{v^2}{\rho}\text{ .... (6)}$$

The two parts of which this expression consists are, evidently, the resolved pressure on the curve produced by the forces X and Y, and the pressure due to the velocity only.

170. *To find the point where the particle will leave the constraining curve.*

For this it is evident that we have only to put $R=0$, as then the motion will be free.

This condition gives us

$$\frac{v^2}{\rho}=Y\cos\theta-X\sin\theta.$$

Now let F be the resultant of X and Y, then if Q be the chord of curvature at P parallel to F, Q is evidently

$$=2\rho\sin FPT=2\rho.\sin(FPX-\theta)$$

$$=2\rho\frac{Y\cos\theta-X\sin\theta}{\sqrt{(X^2+Y^2)}}.$$

Hence, $$\frac{v^2}{2} = \frac{Q}{4}\sqrt{(X^2 + Y^2)}$$

$$= F\frac{Q}{4}.$$

Comparing this with the formula $\frac{1}{2}v^2 = fs$ (§ 77), we see *that the particle will leave the curve at a point where its velocity is such as would be produced by the resultant force then acting on it, if continued constant during its fall from rest through a space equal to $\frac{1}{4}$ of the chord of curvature parallel to that resultant.*

171. The formulæ just given are much simplified when we consider gravity to be the only force acting. Taking in this case the axis of y vertically *upwards*, our forces become

$$X = 0 \text{ and } Y = -g;$$

and the velocity, and the pressure on the curve, are given by

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}V^2 = g(k - y), \text{ if } v = V \text{ when } y = k;$$

$$\text{and } R = g\cos\theta + \frac{v^2}{\rho}.$$

Suppose we change the origin to the point from which the particle's motion is supposed to commence; and take the axis of y vertically *downwards;* we shall evidently have

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}V^2 = gy;$$

and if the particle start from rest

$$\tfrac{1}{2}v^2 = gy.$$

This shews that the velocity depends merely on the distance beneath a horizontal plane through the original position of rest. Hence, whatever be the nature of the curve on which a particle slides under the action of gravity, its motion will always be in the same direction till it rises to the same level as that to the fall from which its velocity is due. If it cannot do so, its motion will be constantly in the same direction; if it can, its velocity will become zero, and the

particle will *then* either come permanently to rest, or return to the point from which it started.

172. *To find the time of a particle's sliding down any arc of a curve, from rest at the upper extremity of the arc.*

Taking the upper extremity as origin and the axis of y vertically downwards; we have

$$\frac{ds}{dt} = v = \sqrt{(2gy)};$$

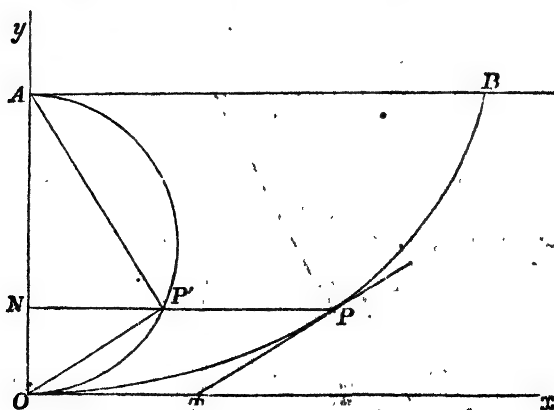
$$\text{and } t_1 = \int_0^{y_1} \frac{\frac{ds}{dy} dy}{\sqrt{(2gy)}} \quad \dots\dots\dots\dots\dots\dots (1)$$

if y_1 be the vertical co-ordinate of the lower extremity of the given arc.

Or, taking the lower point as origin, and axis of y upwards, we have, since in this case v tends to decrease s,

$$t_1 = \int_{y_1}^{0} \frac{-\frac{ds}{dy} dy}{\sqrt{\{2g(y_1 - y)\}}} = \int_0^{y_1} \frac{\frac{ds}{dy} dy}{\sqrt{\{2g(y_1 - y)\}}} \dots\dots (2).$$

173. Ex. *To find the time of descending from rest at any point of an inverted cycloid to the vertex.*



Taking formula (2); since in this case the vertex is origin, and axis that of y, we have from the figure

$$s = OP = 2 \text{ chord } OP' = 2\sqrt{(AO \cdot ON)} = 2\sqrt{(2ay)},$$

if a be the radius of the generating circle.

Hence, $$\frac{ds}{dy} = \sqrt{\frac{2a}{y}};$$

and $$t_1 = \sqrt{\frac{a}{g}} \int_0^{y_1} \frac{dy}{\sqrt{(yy_1 - y^2)}},$$

$$= \left(C + \sqrt{\frac{a}{g}} \operatorname{vers}^{-1} \frac{2y}{y_1}\right)_0^{y_1},$$

$$= \pi \sqrt{\frac{a}{g}};$$

which is independent of y_1, that is, of the point from which the particle begins its descent.

The reason of this remarkable property will be more easily seen if we take the formula for the acceleration in the direction of the arc. We have thus

$$\frac{d^2 s}{dt^2} = -g \sin (P'Ox)$$

(since OP' is parallel to the tangent to the cycloid at P)

$$= -g \sin (OAP')$$

$$= -g \frac{OP'}{OA}$$

$$= -g \frac{s}{4a},$$

or, the acceleration is proportional to the distance from the vertex measured along the cycloid. Comparing this with §§ 83—85, the reason of the above result will be evident.

174. *A particle acted on by gravity moves in an arc of a vertical circle, to determine the motion.*

Taking the vertical diameter as axis of y, and its lower extremity as origin, the equation to the circle is

$$x = \sqrt{(2ay - y^2)}.$$

$$\text{Hence } \frac{ds}{dy} = \frac{a}{\sqrt{(2ay - y^2)}}.$$

$$\text{But } \frac{ds}{dt} = -\sqrt{\{2g(y_1 - y)\}},$$

if we suppose the motion to commence at the point defined by y_1; and therefore

$$\frac{dt}{dy} = -\frac{a}{\sqrt{(2g)}} \frac{1}{\sqrt{\{(y_1 - y)(2ay - y^2)\}}} \quad \text{.........} \quad (1).$$

If we put $y = y_1 \sin^2\theta$, we have for the time of falling through any arc

$$t = -\sqrt{\frac{a}{g}} \int \frac{d\theta}{\sqrt{\left(1 - \frac{y_1}{2a}\sin^2\theta\right)}},$$

an elliptic integral of the first order, whose value for given limits can only be approximated to; except when $y_1 = 2a$, that is, when the velocity is that due to a fall from the highest point of the circle. This case we will soon consider (§ 176).

(1) may be put in the form

$$\frac{dt}{dy} = -\frac{1}{2}\sqrt{\frac{a}{g}} \frac{1}{\sqrt{(y_1 y - y^2)}} \left(1 - \frac{y}{2a}\right)^{-\frac{1}{2}}$$

$$= -\frac{1}{2}\sqrt{\frac{a}{g}} \frac{1}{(y_1 y - y^2)^{\frac{1}{2}}} \left\{1 + \frac{1}{2}\left(\frac{y}{2a}\right) + \frac{1.3}{2.4}\left(\frac{y}{2a}\right)^2\right.$$

$$\left. + \frac{1.3.5}{2.4.6}\left(\frac{y}{2a}\right)^3 + \dots + \frac{1.3\dots(2n-1)}{2.4\dots 2n}\left(\frac{y}{2a}\right)^n + \&c.\right\},$$

each term of which may be integrated separately.

Suppose it be required to determine the time of descent to the lowest point; the limits of y are y_1 and 0. If we notice that

$$\int \frac{y^n dy}{\sqrt{(y_1 y - y^2)}} = \frac{2n-1}{2n} y_1 \int \frac{y^{n-1} dy}{\sqrt{(y_1 y - y^2)}} - \frac{y^{n-1}\sqrt{(y_1 y - y^2)}}{n}:$$

whence $$\int_{y_1}^{0} \frac{y^n dy}{\sqrt{(y_1 y - y^2)}} = \frac{2n-1}{2n} y_1 \int_{y_1}^{0} \frac{y^{n-1} dy}{\sqrt{(y_1 y - y^2)}};$$

while $$\int_{y_1}^{0} \frac{dy}{\sqrt{(y_1 y - y^2)}} = \left(\text{vers}^{-1} \frac{2y}{y_1} + C\right)_{y_1}^{0} = -\pi;$$

we have $$\int_{y_1}^{0} \frac{y^n dy}{\sqrt{(y_1 y - y^2)}} = -\frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n} \pi y_1^n.$$

Hence the time of fall to the lowest point is

$$t_1 = \frac{\pi}{2}\sqrt{\frac{a}{g}} \left[1 + \left(\frac{1}{2}\right)^2 \frac{y_1}{2a} + \left(\frac{1.3}{2.4}\right)^2 \left(\frac{y_1}{2a}\right)^2 + \ldots + \left\{\frac{1.3 \ldots (2n-1)}{2.4 \ldots 2n}\right\}^2 \left(\frac{y_1}{2a}\right)^n + \ldots\right].$$

When the arc of vibration is very small, we have

$$t_1 = \frac{\pi}{2}\sqrt{\frac{a}{g}},$$

and the time of a complete oscillation is

$$2\pi\sqrt{\frac{a}{g}}.$$

The value of t_1 coincides with that in a cycloid, § 173, if we observe that in the cycloid the quantity a is 4 times as great as in the circle.

175. The next approximation gives, as a correction to the period of a quarter oscillation, the expression

$$\frac{\pi}{2}\sqrt{\frac{a}{g}}\,\frac{y_1}{8a}.$$

whose ratio to that period is

$$\frac{y_1}{8a} = (\tfrac{1}{4} \text{ chord semi-angle of oscillation})^2.$$

Thus, if the particle oscillate through an arc whose chord is $\frac{1}{10}$, on each side of the vertical, the time of oscillation given by the formula $\pi\sqrt{\frac{a}{g}}$ will be incorrect by about $\frac{1}{1600}$ of its amount, in defect.

When the particle is supposed to be suspended by a thread without weight, it becomes what is termed a *simple pendulum*. Such a machine can exist only in theory, but Dynamics furnishes us with the means of reducing the calculation of the motion of such a pendulum as we can construct, to that of the simple pendulum. It is evident that by its means we may determine the value of g, if the length of the pendulum, its arc of oscillation, and the number of vibrations it makes in a given time, be known. Since gravity decreases (according to a known law) as we ascend above the Earth's surface, the comparison of the times of vibration of the same pendulum on the top of a mountain and at its base would give approximately the height. One of the most important applications of the pendulum is that made by Newton. It is evident that if the weight of a body be not proportional to its mass, the value of g will be different for different materials. Hence the fact that pendulums of the same length vibrate in equal times at the same place whatever be the matter of which the bob is made, proves, by means of the above formula, the truth of one part of the Law of Gravitation, § 149: viz. that, ceteris paribus, the attraction exerted by one body on another is proportional to the quantity of matter it contains, and independent of its quality.

176. Or we may take the equation for the acceleration along the arc.

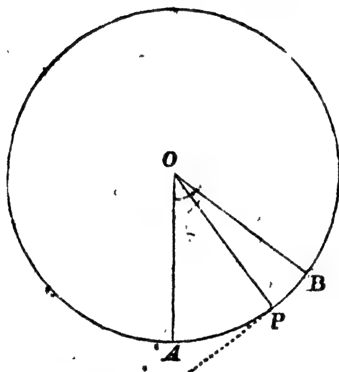
Suppose O to be the center, OA the vertical radius, B the point whence the particle starts with velocity $a\omega$, at time $t=0$; P its position at time t.

Let $AOB=\alpha$, $AOP=\theta$, $OA=a$.

$$\text{Then } \frac{d^2s}{dt^2} = -g \sin\theta.$$

$$\text{But } s = a\theta.$$

$$\text{Hence } \frac{d^2\theta}{dt^2} = -\frac{g}{a}\sin\theta \quad\text{.................(1).}$$



Multiplying by $2\frac{d\theta}{dt}$ and integrating, we have

$$\left(\frac{d\theta}{dt}\right)^2 = C + \frac{2g}{a}\cos\theta.$$

$$\text{But } \frac{d\theta}{dt} = \omega, \text{ when } \theta = \alpha,$$

$$\text{hence } \frac{d\theta}{dt} = \sqrt{\frac{2g}{a}}\sqrt{\left(\cos\theta - \cos\alpha + \frac{a\omega^2}{2g}\right)} \quad\text{......... (2).}$$

This cannot be integrated without elliptic functions unless

$$\frac{a\omega^2}{2g} - \cos\alpha = 1;$$

$$\text{or } a^2\omega^2 = 2ga(1 + \cos\alpha);$$

i.e. unless the velocity of projection at B, be that due to a fall through the difference of altitudes of B and the highest point of the circle.

In this case,

$$\frac{d\theta}{dt} = 2\sqrt{\frac{g}{a}}\cos\frac{\theta}{2}.$$

From this we have

$$\sqrt{\frac{g}{a}}\,t = \log_e C\sqrt{\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}}.$$

But $t=0$, $\theta=\alpha$, together,

therefore $$\sqrt{\frac{g}{a}}\,t = \log_e\sqrt{\frac{\left(1-\sin\frac{\alpha}{2}\right)\left(1+\sin\frac{\theta}{2}\right)}{\left(1+\sin\frac{\alpha}{2}\right)\left(1-\sin\frac{\theta}{2}\right)}}\,\text{.........}(3),$$

which determines the motion completely.

From the remark in § 171, it is evident that, after reaching A, the particle will ascend the other semicircle with a velocity just sufficient to carry it to the highest point; the time, T, at which it will reach that point after leaving A, will be found by putting

$$\theta = \pi,\ \alpha = 0, \text{ in (3).}$$

This gives $$\sqrt{\frac{g}{a}}\,T = \log_e\infty = \infty;$$

or, the particle will continually approach the highest point, but never reach it.

177. *To find the pressure on the circle.*

Suppose R directed outwards from the center, then evidently

$$R = \frac{v^2}{a} + g\cos\theta$$

$$= 2g(\cos\theta - \cos\alpha) + a\omega^2 + g\cos\theta, \qquad \text{by (2)},$$

$$= 3g\cos\theta - 2g\cos\alpha + a\omega^2.$$

Suppose the particle to have been projected from A, with velocity $a\omega$; then $\alpha = 0$;

$$\text{and } R = 3g \cos\theta - 2g + a\omega^2.$$

This expression for R admits of the value zero if

$$a\omega^2 \not> 5g, \text{ or } a\omega \not> \sqrt{(5ga)}.$$

It may happen however that the points thus found may not lie within the arc which the particle passes over.

There are positions of rest (§ 171) when $a\omega \not> 2\sqrt{(ga)}$. Now, in order that the points where $R = 0$ may lie within the limits of oscillation, the value of $\cos\theta$, for the former, must not be less than that for the latter;

$$\text{or, } \frac{2g - a\omega^2}{3g} \not< \frac{2g - a\omega^2}{2g}.$$

This condition can only be satisfied by $2g - a\omega^2$ vanishing or becoming negative; that is, by

$$a\omega \not< \sqrt{(2ga)}.$$

Hence, if the velocity of projection from the lowest point do not fall short of $\sqrt{(2ga)}$, and do not exceed $\sqrt{(5ga)}$, there will be a point in the path at which $R = 0$; and if the particle be moving on the concave side of a smooth circle, or be attached by a string to a fixed point, the circular motion will cease at that point; the particle will fall off the circle in the one case, and the string will cease to be stretched in the other.

Beyond these limits it is evident that we shall have, for velocity of projection $> \sqrt{(5ga)}$ continuous revolution in the circle, and for velocity of projection $< \sqrt{(2ga)}$ oscillations about the lowest point.

Also by what we have before shewn, if the particle be constrained by a circular tube, it will oscillate if the velocity at the lowest point is less than $2\sqrt{(ga)}$: if that velocity be equal to $2\sqrt{(ga)}$ the particle will reach the highest point after

the lapse of an infinite time; and if greater than $2\sqrt{(ga)}$ it will revolve continuously.

178. *A particle falls from rest at a height* k *down the semicubical parabola whose equation is* $ax^2 = y^3$, *the axis of* y *being vertical; to determine the motion.*

Here, $\frac{ds}{dt} = -\sqrt{\{2g(k-y)\}}$, since gravity tends to diminish s.

Also $\frac{ds}{dy} = \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}} = \sqrt{\left(1 + \frac{9y^4}{4a^2x^2}\right)} = \sqrt{\left(1 + \frac{9y}{4a}\right)}$.

$$\text{Hence, } \frac{dt}{dy} = -\sqrt{\frac{4a+9y}{8ga(k-y)}};$$

and the time of fall to a point where $y = l$ is

$$t_1 = \frac{1}{\sqrt{(8ga)}}\int_l^k \sqrt{\frac{4a+9y}{k-y}}\,dy.$$

Let $\frac{\theta^2}{9} = \frac{k-y}{4a+9y}$, the limits of θ are $\sqrt{\frac{9(k-l)}{4a+9l}}$ and 0;

$$\text{and therefore } t_1 = \frac{9k+4a}{3\sqrt{(8ga)}}\int_0^{\sqrt{\frac{9(k-l)}{4a+9l}}} \frac{2d\theta}{(1+\theta^2)^2}.$$

Put $\theta = \tan\phi$, limits are $\tan^{-1}\sqrt{\frac{9(k-l)}{4a+9l}}$ and 0, and

$$t_1 = \frac{9k+4a}{3\sqrt{(8ga)}}\int_0^{\tan^{-1}\sqrt{\frac{9(k-l)}{4a+9l}}} 2d\phi\cos^2\phi$$

$$= \frac{9k+4a}{3\sqrt{(8ga)}}\left\{\tan^{-1}\sqrt{\frac{9(k-l)}{4a+9l}} + \frac{\sqrt{\frac{9(k-l)}{4a+9l}}}{1+\frac{9(k-l)}{4a+9l}}\right\};$$

which determines t_1 for any values of k and l.

If the time of fall to the cusp at the origin be required, $l=0$, and

$$T=\frac{9k+4a}{3\sqrt{(8ga)}}\left\{\tan^{-1}\frac{3}{2}\sqrt{\frac{k}{a}}+\frac{6\sqrt{(ak)}}{4a+9k}\right\}.$$

179. *A particle acted on by gravity is projected, from the vertex, along a smooth parabola whose axis is vertical and vertex upwards; to determine the motion and the pressure on the curve.*

Let $x^2=4ay$ be the equation, the axis of y being vertically downwards, and the vertex the origin. Then

$$\left(\frac{ds}{dt}\right)^2=v^2=V^2+2gy,$$

where V is the given velocity at the vertex. Suppose it due to a height l, then

$$\left(\frac{ds}{dt}\right)^2=2g(l+y) \quad\text{..........................}(1).$$

$$\text{Now }\left(\frac{dy}{ds}\right)^2=\frac{y}{a+y},$$

by the equation to the curve.

$$\text{Hence }\left(\frac{dy}{dt}\right)^2=\frac{2gy(l+y)}{a+y},$$

and, if t_1 be the time of fall to a depth k,

$$t_1=\frac{1}{\sqrt{(2g)}}\int_0^k\sqrt{\frac{a+y}{y(l+y)}}\,dy,$$

which is thus determined.

For the pressure on the curve, supposing it positive when *from* the axis,

$$R = \frac{v^2}{\rho} - g\frac{dx}{ds}$$

$$= \frac{2g(l+y)}{2a\left(\frac{a+y}{a}\right)^{\frac{3}{2}}} - g\left(\frac{a}{a+y}\right)^{\frac{1}{2}}$$

$$= g\,\frac{(l-a)\,a^{\frac{1}{2}}}{(a+y)^{\frac{3}{2}}}.$$

If $l > a$, R is positive, or the particle will move on the concave side of the curve; if $l < a$, R is negative, and the particle moves on the convex side. In each of these cases the pressure is inversely as the $\frac{3}{2}$th power of the distance below the directrix.

If $l = a$, that is if the velocity of projection at the vertex be that due to a fall from the directrix, R is zero the whole way, or the particle moves freely, as we might have inferred from the results of Chap. IV.

180. *To find a curve such that a particle under the action of gravity will descend any arc of it from a given point, in the same time as it takes to descend the chord of that arc.*

Take the vertical through the given point as the initial line, then if ρ, θ be the polar co-ordinates of a point in the curve, the given point being pole, the conditions of the problem give at once

$$\int_{\theta_0}^{\theta} \frac{\frac{ds}{d\theta}\,d\theta}{\sqrt{(2g\rho\cos\theta)}} = \sqrt{\frac{2\rho}{g\cos\theta}},$$

θ_0 being the inclination to the vertical of the tangent at the point of departure.

Or, differentiating with respect to θ,

$$\frac{\frac{ds}{d\theta}}{\sqrt{(2\rho\cos\theta)}} = \frac{\sqrt{2}}{2}\left\{\frac{\frac{d\rho}{d\theta}}{\sqrt{(\rho\cos\theta)}} + \frac{\sqrt{\rho}\sin\theta}{(\cos\theta)^{\frac{3}{2}}}\right\}.$$

$$\text{Hence } \frac{ds}{d\theta} = \frac{d\rho}{d\theta} + \rho \tan\theta.$$

$$\text{But } \left(\frac{ds}{d\theta}\right)^2 = \rho^2 + \left(\frac{d\rho}{d\theta}\right)^2.$$

Eliminating s between these equations, and reducing, we obtain

$$2\frac{d\rho}{\rho} = 2\frac{\cos 2\theta\, d\theta}{\sin 2\theta},$$

whose integral is

$$\log_e \rho^2 = \log_e C \sin 2\theta,$$

$$\text{that is, } \rho^2 = a^2 \sin 2\theta,$$

the Lemniscate of Bernoulli, the node being the pole, and a tangent at that point the initial line.

181. *To find a curve such that if a particle, acted on by gravity, fall down it through a vertical space* h, *starting from the vertex with velocity due to a height* h, *the time of fall will be independent of* h.

Let the axis of y be vertically downwards, then evidently the time required is

$$t_1 = \int_0^h \frac{\frac{ds}{dy}dy}{\sqrt{\{2g(h+y)\}}}.$$

Let $\frac{ds}{dy} = \phi(y)$ be the required differential equation to the curve,

$$t_1 = \frac{1}{\sqrt{(2g)}}\int_0^h \frac{\phi(y)\,dy}{\sqrt{(h+y)}}$$

$$= \frac{1}{\sqrt{(2g)}}\int_0^1 \frac{\phi(hz)\,h^{\frac{1}{2}}dz}{\sqrt{(1+z)}}, \text{ if } y = hz.$$

Now $\frac{dt_1}{dh}$ is to be identically zero, hence we have

$$hz\,\phi'(hz) = -\tfrac{1}{2}\phi(hz),$$

$$\text{or}\quad \frac{\phi'(y)\,dy}{\phi(y)} = -\frac{dy}{2y},$$

$$\text{whence}\quad \phi(y) = Cy^{-\frac{1}{2}} = \frac{ds}{dy},$$

which shews the curve to be a cycloid, whose base is horizontal and vertex upwards.

182. *Two points being given, which are neither in a vertical nor in a horizontal line, to find the curve joining them, down which a particle sliding under the action of gravity, and starting from rest at the higher, will reach the other in the least possible time.*

The curve must evidently lie in the vertical plane passing through the points. For suppose it not to lie in that plane, project it on the plane, and call corresponding elements of the curve and its projection σ and σ'. Then if a particle slide down the projected curve its velocity at σ' will be the same as the velocity in the other at σ. But σ is *never less* than σ', and is generally greater. Hence the time through σ' is generally less than that through σ, and *never greater*. That is, the whole time of falling through the projected curve is less than that through the curve itself. Or the required curve lies in the vertical plane through the points.

Taking the axes of x and y, horizontal, and vertically downwards, respectively, from the starting point; if x_0 be the abscissa of the other point, the time of descent will be

$$t_0 = \int_0^{x_0} \frac{\frac{ds}{dx}\,dx}{\sqrt{(2gy)}};\quad \text{or, writing } \frac{dy}{dx} = p,$$

$$t_0 = \int_0^{x_0} \frac{\sqrt{(1+p^2)}}{\sqrt{(2gy)}}\,dx.$$

Applying the rules of the Calculus of Variations, we have, since V or $\dfrac{\sqrt{(1+p^2)}}{\sqrt{y}}$ is a function of y and p, the condition for a minimum,

$$V = p\left(\frac{dV}{dp}\right) + C,$$

the differential coefficient being partial.

This gives $\frac{\sqrt{(1+p^2)}}{\sqrt{y}} = \frac{p^2}{\sqrt{y}\sqrt{(1+p^2)}} + C,$

$$\text{or } \sqrt{y}\sqrt{(1+p^2)} = \frac{1}{C} = \sqrt{a} \text{ suppose.}$$

$$\text{Hence } \frac{ds}{dy} = \frac{\sqrt{(1+p^2)}}{p} = \sqrt{\frac{a}{a-y}},$$

the differential equation to a cycloid, the origin being a cusp and the base the axis of x.

This is a problem celebrated in the history of Dynamics. The cycloid has received on account of this property the name of Brachistochrone. Farther on we propose to investigate the nature and some of the properties of Brachistochrones for other forces besides gravity. For an investigation not involving the Calculus of Variations see Appendix.

183. *To find the curve down which if a particle, projected with a given velocity, slide under the action of gravity, it will descend equal vertical spaces in equal times.*

Here we have, taking the axis of x horizontal, and that of y vertically downwards,

$$\frac{ds}{dt} = \sqrt{(2gy)};$$

if the velocity is that due to a fall from the axis of x.

Also by condition $\frac{dy}{dt} = \text{const.} = \sqrt{(2gh)}$, suppose.

$$\text{Hence } \frac{ds}{dy} = \sqrt{\frac{y}{h}},$$

$$\frac{dx}{dy} = \sqrt{\frac{y-h}{h}};$$

$$\text{or, } x + C = \frac{2}{3}\left\{\frac{(y-k)^3}{k}\right\}^{\frac{1}{2}};$$

the semicubical parabola.

If the horizontal velocity is to be constant, we have

$$\frac{dx}{dt} = \sqrt{(2gk)}; \quad \therefore \frac{ds}{dx} = \sqrt{\frac{y}{k}}, \text{ or } \frac{dy}{dx} = \sqrt{\frac{y-k}{k}},$$

and therefore $\quad 2\sqrt{k}\sqrt{(y-k)} = x + C;$

a parabola with its axis vertical and vertex upwards; as indeed we might have forcseen from the results of Chap. IV.

184. *A particle moves on a smooth plane curve under the action of a force directed to a fixed center in the plane of the curve; to determine the motion.*

Let $r = f(\theta)$ be the polar equation of the constraining curve about the center of force as pole, and let $P = \phi(r)$ be the central *repulsive* force on a particle whose distance from the center is r.

Resolving along the tangent at any point,

$$\frac{d^2s}{dt^2} = P\frac{dr}{ds} \text{ .... (1).}$$

Hence, $$\left(\frac{ds}{dt}\right)^2 = v^2 = C + 2\int\phi(r)\,dr \text{ .... (2).}$$

Equation (2) contains the complete solution of the problem so far as the motion is concerned; since, by means of the equation to the curve, either r or s may be eliminated from it, and if the resulting differential equation be integrable, it will give s or r in terms of t.

For the pressure on the curve. Resolving along the normal at any point, ρ being the radius of curvature, we have

$$\frac{v^2}{\rho} + P\frac{r\,d\theta}{ds} = R, \text{ .... (3),}$$

an expression which by means of the foregoing equations will give R in terms of t or r.

Hence the solution is complete.

185. *A particle, initially at rest at a point of the logarithmic spiral* $r = a\epsilon^{n\theta}$ *whose radius vector is* b, *moves on the curve under the action of an attracting center of force* $\propto$ *distance, situated at the pole; to determine the motion.*

$$\text{Here } \frac{d^2s}{dt^2} = -\mu r \frac{dr}{ds};$$

$$\text{therefore } \left(\frac{ds}{dt}\right)^2 = C - \mu r^2,$$

$$\text{and } 0 = C - \mu b^2,$$

$$\text{hence } \frac{ds}{dt} = \sqrt{\{\mu (b^2 - r^2)\}}.$$

$$\text{But } \frac{ds}{dt} = \sqrt{\left\{\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2\right\}}$$

$$= \frac{dr}{dt}\sqrt{\left(1 + \frac{1}{n^2}\right)}.$$

$$\text{We have therefore } \frac{dr}{\sqrt{(b^2 - r^2)}} = \frac{n\sqrt{\mu}}{\sqrt{(1 + n^2)}}\, dt,$$

$$\text{whence } r = b \cos\left\{\frac{n\sqrt{\mu t}}{\sqrt{(1 + n^2)}} + \beta\right\}.$$

At time $t = 0$, $b = b\cos\beta$; which gives $\beta = 0$,

$$\text{and finally } r = b\cos\frac{n\sqrt{\mu t}}{\sqrt{(1 + n^2)}};$$

which determines the position of the particle at any time.

When it reaches the pole $r = 0$; the required interval is therefore

$$t' = \frac{\pi\sqrt{(1 + n^2)}}{2n\sqrt{\mu}}.$$

For the pressure on the curve,

$$R = \frac{v^2}{\rho} - \mu r \frac{rd\theta}{ds}$$

$$= \frac{\mu (b^2 - r^2)}{r\sqrt{(1+n^2)}} - \frac{\mu r}{\sqrt{(1+n^2)}}$$

$$= \frac{\mu}{\sqrt{(1+n^2)}} \frac{b^2 - 2r^2}{r}.$$

Hence the pressure is towards the pole when the motion commences, becomes zero when the particle's distance from the pole is diminished in the ratio of $\frac{1}{\sqrt{2}}$, and then is directed from the pole for the rest of the motion.

186. *When the constraining curve is one of double curvature.*

All we know directly about R is that it is perpendicular to the tangent line at any point.

Resolve then the given forces acting upon the particle into three, one, S, along the tangent, which in all cases in nature will be a function of x, y, z and therefore of s; another, T, in the line of intersection of the normal and osculating planes (or radius of absolute curvature); and the third, P, perpendicular to each of the other two.

Let the resolved parts of R in the directions of T and P be R_1, R_2. Then the acceleration along the tangent is $\frac{d^2s}{dt^2}$, and therefore

$$\frac{d^2s}{dt^2} = S \text{ .... (1).}$$

This equation together with the two to the curve is sufficient to determine the *motion* completely.

Now the particle at any point of its path may be considered as moving in the osculating plane. Hence, by our investigation for motion on a plane curve, § 169, if ρ be the radius of absolute curvature, v the velocity,

$$R_1 = \frac{v^2}{\rho} - T \text{ .... } (2),$$

T being considered positive when it acts towards the center of absolute curvature.

Now R_2 is the force which prevents P's withdrawing the particle from the osculating plane; and therefore

$$R_2 = -P \text{ .... } (3),$$

(2) and (3) give the resolved parts of the pressure on the curve.

Also $R = \sqrt{(R_1^2 + R_2^2)}$, and makes an angle $= \tan^{-1}\left(\frac{R_2}{R_1}\right)$ with the osculating plane.

187. In Art. 173 we arrived at the remarkable property of the cycloid, that a particle falling under the action of gravity from rest at any point of the curve reaches the lowest point in the same time, whatever be the point of the curve from which it starts. *Let us find for what forces a similar property is possessed by any other given curve.*

Let the forces resolved along the curve have a component $= \phi'(s)$, where s is the distance at any instant from some fixed point: then,

$$\frac{d^2 s}{dt^2} = S = -\phi'(s) \text{ .... } (1);$$

and if the particle starts at a distance k from the fixed point, the velocity $= 0$ when $s = k$. Hence the corrected integral of (1) is

$$\left(\frac{ds}{dt}\right)^2 = 2\{\phi(k) - \phi(s)\};$$

and we have
$$\sqrt{2}\tau = \int_k^0 \frac{-ds}{\{\phi(k) - \phi(s)\}^{\frac{1}{2}}};$$

if τ be the time of fall to the fixed point, which is by hypothesis to be independent of k.

Put $s=kz$, the limits of z are 1 and 0, and

$$\sqrt{2}\tau=\int_0^1 \frac{kdz}{\{\phi(k)-\phi(kz)\}^{\frac{1}{2}}};$$

and that this may be independent of k,

$$\phi(k)-\phi(kz)=k^2 f(z);$$

which gives $$\frac{\phi(k)}{k^2}-z^2\frac{\phi(kz)}{k^2z^2}=f(z).$$

Hence, by inspection, $$\frac{\phi(k)}{k^2}=C'+\frac{C''}{k^2}\text{..............(2).}$$

Or thus, $$\sqrt{2}\frac{d\tau}{dk}=\int_0^1 \frac{\{\phi(k)-\phi(kz)\}^{\frac{1}{2}}-\frac{k}{2}\frac{\phi'(k)-z\phi'(kz)}{\{\phi(k)-\phi(kz)\}^{\frac{1}{2}}}}{\phi(k)-\phi(kz)}dz,$$

which must be identically equal to nothing.

Hence $$\{\phi(k)-\frac{k}{2}\phi'(k)\}-\{\phi(kz)-\frac{kz}{2}\phi'(kz)\}=0$$

identically, which can only be the case if

$$\phi(x)-\frac{x}{2}\phi'(x)=C'';$$

(or if $\phi(x)=$ constant; which we evidently need not consider, as in this case there would be no acceleration.)

Hence $$-2\frac{\phi(x)}{x^3}+\frac{\phi'(x)}{x^2}=-\frac{2C''}{x^3},$$

which gives, as above, $$\frac{\phi(x)}{x^2}=C'+\frac{C''}{x^2}\text{..................(2),}$$

or $$\phi(x)=C'x^2+C'',$$

and $$\phi'(x)=Cx.$$

Hence, by (1), $\frac{d^2s}{dt^2} = S = -Cs$ ....(3),

that is, the resolved force along the curve must be proportional to the arcual distance from the fixed point.

188. We might have arrived at the same conclusion, but not quite so satisfactorily, thus,

$$\sqrt{2}\tau = \int_0^k \frac{ds}{\{\phi(k)\}^{\frac{1}{2}}} \left[1 + \frac{1}{2}\frac{\phi(s)}{\phi(k)} + \dots + \frac{\lfloor 2n}{2^{2n}(\lfloor n)^2}\left\{\frac{\phi(s)}{\phi(k)}\right\}^n + \dots\right].$$

Now the condition that the $(n+1)^{\text{th}}$ term when integrated between the limits should not contain k is that

$$\int_0^k \frac{\{\phi(s)\}^n ds}{\{\phi(k)\}^{n+\frac{1}{2}}} \text{ should be independent of } k.$$

This can only be the case if $\{\phi(k)\}^{\frac{1}{2}}$, and of course also $\{\phi(s)\}^{\frac{1}{2}}$, be of the same dimensions as ds, and therefore as s.

$$\text{Hence take } \{\phi(s)\}^{\frac{1}{2}} = C''s,$$

$$\text{or } \phi(s) = C's^2;$$

and we have $\phi'(s) = Cs$, as before.

Hence, if X, Y, Z be the impressed forces,

$$X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} = -Cs$$

is the condition they must satisfy at every point x, y, z of the given curve. For such forces the given curve is said to be a *Tautochrone.*

By equation (3) § 187, the time of descent is

$$\tau = \frac{\pi}{2\sqrt{C}}. \quad \text{Hence } C = \frac{\pi^2}{4\tau^2}.$$

189. *To find the Brachistochrone for a particle subjected to the action of any forces which make* Xdx + Ydy + Zdz *a complete differential.*

Generally $t = \int \frac{ds}{v}$, between proper limits, is to be a minimum; and therefore, taking its variation,

$$\delta t = \int \frac{v\delta ds - ds\delta v}{v^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But, here, $\frac{1}{2} v^2 = \int (Xdx + Ydy + Zdz)$;

which gives $v\delta v = X\delta x + Y\delta y + Z\delta z$,

$$\text{or } ds\delta v = (X\delta x + Y\delta y + Z\delta z)\, dt \ldots\ldots\ldots\ldots\ldots (2).$$

$$\text{Again } ds^2 = dx^2 + dy^2 + dz^2,$$

$$\text{and } \frac{ds}{dt}\delta ds = v\delta ds = \frac{dx}{dt}\delta dx + \frac{dy}{dt}\delta dy + \frac{dz}{dt}\delta dz \ldots\ldots\ldots (3).$$

Hence (1) becomes, by (2) and (3), and since d and δ follow the commutative law,

$$0 = \int \frac{1}{v^2}\left(\frac{dx}{dt} d\delta x + \frac{dy}{dt} d\delta y + \frac{dz}{dt} d\delta z\right)$$

$$- \int \frac{1}{v^2}(X\delta x + Y\delta y + Z\delta z)\, dt$$

$$= \left[\frac{1}{v^2}\left(\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right)\right]$$

$$- \left\{\frac{1}{v^2}\left(\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right)\right\}$$

$$- \int dt \left[\left\{\frac{d}{dt}\left(\frac{1}{v^2}\frac{dx}{dt}\right) + \frac{X}{v^2}\right\}\delta x + \left\{\frac{d}{dt}\left(\frac{1}{v^2}\frac{dy}{dt}\right) + \frac{Y}{v^2}\right\}\delta y\right.$$

$$\left. + \left\{\frac{d}{dt}\left(\frac{1}{v^2}\frac{dz}{dt}\right) + \frac{Z}{v^2}\right\}\delta z\right],$$

by integrating the first term by parts. The integrated terms in [ ] belong to the superior, those in { } to the inferior, limit.

But, if the terminal points are given, we have at both limits

$$\delta x = 0,\ \delta y = 0,\ \delta z = 0,$$

and therefore the terms independent of the integral sign vanish. In order that the integral may be identically zero, we must have

$$\frac{d}{dt}\left(\frac{1}{v^2}\frac{dx}{dt}\right) + \frac{X}{v^2} = 0 \text{ .... } (4),$$

with similar expressions in y and z. The elimination of t, and v or $\frac{ds}{dt}$, from these equations will give us two differential equations to the curve required, the forces X, Y, Z being by hypothesis functions of x, y, z only.

But without getting rid of v we may prove two properties common to all such Brachistochrones.

Eliminating t from (4) we have

$$v\frac{d}{ds}\left(\frac{1}{v}\frac{dx}{ds}\right) + \frac{X}{v^2} = 0,$$

$$\text{or } v^2\frac{d^2x}{ds^2} - v\frac{dv}{ds}\frac{dx}{ds} + X = 0 \text{ .... } (5),$$

with similar expressions in y and z.

Multiplying these in order by λ, μ, ν and adding; if we take λ, μ, ν such that

$$\left.\begin{aligned} \lambda\frac{d^2x}{ds^2} + \mu\frac{d^2y}{ds^2} + \nu\frac{d^2z}{ds^2} &= 0 \\ \lambda\frac{dx}{ds} + \mu\frac{dy}{ds} + \nu\frac{dz}{ds} &= \end{aligned}\right\} \text{ .... } (6),$$

we shall have also

$$\lambda X + \mu Y + \nu Z = 0 \text{ .... } (7).$$

Now (6) shews that the line whose direction cosines are as λ, μ, ν is perpendicular to the radius of absolute curvature of the path, and also to the tangent; that is, it is normal to the osculating plane. Also by (7) the same line is perpendicular to the resultant of X, Y, Z.

Hence, *the osculating plane at any point contains the resultant of the impressed forces.*

Again, if ρ be the radius of absolute curvature,

$$\rho = \left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right\}^{-\frac{1}{2}},$$

and its direction cosines are

$$\rho\frac{d^2x}{ds^2},\ \rho\frac{d^2y}{ds^2},\ \rho\frac{d^2z}{ds^2};$$

therefore, multiplying equations (5) by

$$\frac{d^2x}{ds^2},\ \frac{d^2y}{ds^2},\ \frac{d^2z}{ds^2},$$

and adding, noting that, since

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1,$$

we have

$$\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2} = 0,$$

we obtain the equation

$$\frac{v^2}{\rho} = -\left(X\rho\frac{d^2x}{ds^2} + Y\rho\frac{d^2y}{ds^2} + Z\rho\frac{d^2z}{ds^2}\right) \quad\ldots\ldots\ldots\ldots\ (8),$$

which expresses that *the portion of the pressure due to the velocity is equal to that produced by the impressed forces.*

190. If the terminal points are not definitely assigned (if, for instance, it be required to find the line of swiftest descent from one given curve to another) we have no longer

$$\delta x = 0,\ \delta y = 0,\ \delta z = 0$$

at the limits; but, with the requisite modifications, the process in § 189 enables us to find the proper conditions in any case. These questions, however, belong rather to Calculus of Variations than to Kinetics.

Thus, suppose that the final point of the path is to lie on

$$F(x, y, z) = 0,$$

we have

$$\frac{dF}{dx}\delta x + \frac{dF}{dy}\delta y + \frac{dF}{dz}\delta z = 0 \quad\dots\dots\dots\dots\dots\dots (1).$$

Also that [ ] may vanish, which is necessary in order that δt may be zero, we must have

$$\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z = 0 \quad\dots\dots\dots\dots\dots\dots (2).$$

Now the only relation between δx, δy and δz is (1), to which (2) must therefore be equivalent: hence

$$\frac{dx}{dt} : \frac{dy}{dt} : \frac{dz}{dt} :: \frac{dF}{dx} : \frac{dF}{dy} : \frac{dF}{dz}.$$

These equations show that the moving particle meets the terminal surface at right angles. A similar condition is easily seen to hold if the initial point of the path is also to lie on a given surface, provided the whole energy be given and the given surface be an *equipotential* one. If it be not equipotential, terms depending on δx_0, δy_0, δz_0, will appear in the integral and must be taken along with { }.

If a terminal point is to lie in a given *curve* the condition is to be determined in a similar manner.

191. *A particle moves under the action of given forces on a given smooth surface; to determine the motion, and the pressure on the surface.*

Let

$$F(x, y, z) = 0 \quad\dots\dots\dots\dots\dots\dots\dots\dots\dots (1),$$

be the equation to the surface, R the force acting in the normal to the surface, which is the only effect of the constraint. Then if λ, μ, ν be its direction cosines, we know that

$$\lambda = \frac{\left(\frac{dF}{dx}\right)}{\sqrt{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2\right\}}} \quad\dots\dots\dots\dots (2),$$

with similar expressions for μ and ν; the differential coefficients being partial.

If X, Y, Z be the impressed forces, our equations of motion are, evidently,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= X + R\lambda \\ \frac{d^2y}{dt^2} &= Y + R\mu \\ \frac{d^2z}{dt^2} &= Z + R\nu \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (3).$$

Multiplying equations (3) respectively by

$$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt},$$

and adding, we obtain

$$\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2} = \frac{1}{2}\frac{d(v^2)}{dt}$$

$$= X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt} \dots\dots\dots\dots\dots\dots (4).$$

R disappears from this equation, for its coefficient is

$$\lambda\frac{dx}{dt} + \mu\frac{dy}{dt} + \nu\frac{dz}{dt},$$

and vanishes, because the line whose direction cosines are proportional to $\frac{dx}{dt}$, &c. being the tangent to the path, is perpendicular to the normal to the surface.

If we suppose X, Y, Z to be such forces as occur in nature, (Chap. II.) the integral of (4) will be of the form,

$$v^2 = \phi(x, y, z) + C \dots\dots\dots\dots\dots\dots (5),$$

and the velocity at any point will depend only on the initial circumstances of projection, and not on the form of the path pursued.

To find R, multiply equations (3) in order by λ; μ, ν, add, and observe that $\lambda^2+\mu^2+\nu^2=1$. We thus obtain

$$\lambda\frac{d^2x}{dt^2}+\mu\frac{d^2y}{dt^2}+\nu\frac{d^2z}{dt^2}=X\lambda+Y\mu+Z\nu+R.$$

Now, since $\quad \dfrac{d^2x}{dt^2}=\dfrac{d^2x}{ds^2}\left(\dfrac{ds}{dt}\right)^2+\dfrac{dx}{ds}\dfrac{d^2s}{dt^2}$, &c.

$$\lambda\frac{d^2x}{dt^2}+\mu\frac{d^2y}{dt^2}+\nu\frac{d^2z}{dt^2}=\left(\frac{ds}{dt}\right)^2\left\{\lambda\frac{d^2x}{ds^2}+\mu\frac{d^2y}{ds^2}+\nu\frac{d^2z}{ds^2}\right\};$$

$$\text{for, evidently, } \lambda\frac{dx}{ds}+\mu\frac{dy}{ds}+\nu\frac{dz}{ds}=0.$$

But, if ρ be the radius of curvature of the normal section through δs, ρ_1 the radius of absolute curvature of the path, we have, by Meunier's Theorem,

$$\rho\left(\lambda\rho_1\frac{d^2x}{ds^2}+\mu\rho_1\frac{d^2y}{ds^2}+\nu\rho_1\frac{d^2z}{ds^2}\right)=\rho_1.$$

Hence $\quad \lambda\dfrac{d^2x}{dt^2}+\mu\dfrac{d^2y}{dt^2}+\nu\dfrac{d^2z}{dt^2}=\dfrac{v^2}{\rho}$,

and the above equation becomes

$$\frac{v^2}{\rho}=X\lambda+Y\mu+Z\nu+R,$$

which gives the normal pressure on the surface.

192. *To find the curve which the particle describes on the surface.*

For this purpose we must eliminate R from equations (3). The result is

$$\frac{\dfrac{d^2x}{dt^2}-X}{\lambda}=\frac{\dfrac{d^2y}{dt^2}-Y}{\mu}=\frac{\dfrac{d^2z}{dt^2}-Z}{\nu}\ldots\ldots\ldots\ldots (6),$$

two equations, between which if t be eliminated, the result is the differential equation to a second surface intersecting the first in the curve described.

193. So far for the general case, let us now make particular hypotheses.

If there be no impressed forces on the particle, we have by (5), $v^2 = C$, and equations (6) become, since in this case

$$\frac{d^2x}{dt^2} = \frac{d^2x}{ds^2}\left(\frac{ds}{dt}\right)^2 = C\frac{d^2x}{ds^2}, \text{ \&c. \&c.,}$$

$$\frac{\frac{d^2x}{ds^2}}{\lambda} = \frac{\frac{d^2y}{ds^2}}{\mu} = \frac{\frac{d^2z}{ds^2}}{\nu}.$$

Now $\frac{d^2x}{ds^2}$, &c. are proportional to the direction cosines of the radius of absolute curvature of the path; λ, μ, ν are those of the normal to the surface. Hence those lines coincide, or the normal to the surface lies in the osculating plane to the path.

But this is the property of the longest or shortest line joining two points on a surface, hence we have the following,

If a particle, subject to no forces, move from one point to another of a smooth surface, the length of the path described will be a maximum or minimum.

This result will be afterwards deduced from a different principle (Chap. IX.).

194. *A particle moves on a surface of revolution, the only force acting being gravity parallel to the axis of the surface; to determine the motion.*

Take the axis of the surface as that of z, the equation may be written

$$F(x, y, z) = f\{\sqrt{(x^2 + y^2)}\} - z = 0.$$

This may be put in the form

$$f(\rho) - z = 0,$$

if ρ be the distance of any point in the surface from the axis.

Equations (6) become

$$\frac{\frac{d^2x}{dt^2}}{f'(\rho)\frac{x}{\rho}} = \frac{\frac{d^2y}{dt^2}}{f'(\rho)\frac{y}{\rho}} = \frac{\frac{d^2z}{dt^2} - g}{-1} \quad\ldots\ldots\ldots\ldots (7).$$

The first two equal terms give us, for the motion referre to a plane perpendicular to the axis, the equation

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0.$$

But if θ be the angle between the plane containing ρ an the axis of z, and a fixed plane through that axis; we se (§§ 22, 127) that this is equivalent to

$$\rho^2\frac{d\theta}{dt} = \text{const.} = h \quad\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Now $\frac{dz}{dt} = f'(\rho)\frac{d\rho}{dt} = f'(\rho)\frac{d\rho}{d\theta}\frac{d\theta}{dt} = \frac{hf'(\rho)}{\rho^2}\frac{d\rho}{d\theta}$.

And therefore $\frac{d^2z}{dt^2} = \frac{h^2}{\rho^2}\frac{d}{d\theta}\left\{\frac{f'(\rho)}{\rho^2}\frac{d\rho}{d\theta}\right\}$.

But, in equations (7), multiply the numerator and denominator of the first fraction by x, and those of the seconc by y; then add their numerators and denominators to forn those of a new fraction. It will of course be equal to eithe of the others, and therefore to the third fraction in (7) This gives

$$\frac{x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2}}{\rho f'(\rho)} = g - \frac{d^2z}{dt^2} \quad\ldots\ldots\ldots\ldots\ldots (9).$$

Now by differentiating the equation $x^2 + y^2 = \rho^2$, we obtain

$$x\frac{dx}{dt} + y\frac{dy}{dt} = \rho\frac{d\rho}{d\theta}\frac{d\theta}{dt} = \frac{h}{\rho}\frac{d\rho}{d\theta};$$

And, by a second differentiation,

$$x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\frac{h^2}{\rho^2}\frac{d}{d\theta}\left(\frac{1}{\rho}\frac{d\rho}{d\theta}\right),$$

and (9) becomes

$$\frac{\frac{h^2}{\rho^2}\frac{d}{d\theta}\left(\frac{1}{\rho}\frac{d\rho}{d\theta}\right)-\frac{h^2}{\rho^4}\left\{\rho^2+\left(\frac{d\rho}{d\theta}\right)^2\right\}}{\rho f'(\rho)}=g-\frac{h^2}{\rho^2}\frac{d}{d\theta}\left\{\frac{f'(\rho)}{\rho^2}\frac{d\rho}{d\theta}\right\};$$

or, $$\rho\frac{d^2\rho}{d\theta^2}-2\left(\frac{d\rho}{d\theta}\right)^2-\rho^2=\frac{\rho^5 f'(\rho)}{h^2}\left[g-\frac{h^2}{\rho^2}\frac{d}{d\theta}\left\{\frac{f'(\rho)}{\rho^2}\frac{d\rho}{d\theta}\right\}\right],$$

the differential equation to the projection of the path on the plane of xy. If we omit the term containing g, we see, by § 193, that the above equation will represent the projection on xy of a geodetic line on the given surface.

195. *Suppose the motion to take place in a spherical bowl; or, more simply, let the particle be suspended by a string from a fixed point.*

This is the general case of the *Simple Pendulum*.

Let us take the center as origin, and the axis of z vertically downwards.

Then $$F(x, y, z)=x^2+y^2+z^2-a^2=0,$$

and the equations of motion are

$$\frac{d^2x}{dt^2}=-R\frac{x}{a},$$

$$\frac{d^2y}{dt^2}=-R\frac{y}{a},$$

$$\frac{d^2z}{dt^2}=g-R\frac{z}{a}.$$

Hence, $$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=C+2gz$$

$$=V^2-2g(k-z)\;......(1),$$

if V and k be the initial values of v and z.

$$\text{But } x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0;$$

$$\text{or, } x\frac{dy}{dt} - y\frac{dx}{dt} = h \dots\dots\dots\dots\dots (2).$$

$$\text{Also } x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = 0 \dots\dots\dots\dots (3),$$

by the equation to the surface.

Hence, eliminating $\frac{dx}{dt}$ and $\frac{dy}{dt}$ from (1), (2), (3), we have

$$t = \int \frac{a\,dz}{\sqrt{[(a^2 - z^2)\{V^2 - 2g(k - z)\} - h^2]}} \dots\dots (4),$$

an elliptic function which, if it were integrable in finite terms, would give z, and consequently x and y, in terms of t.

196. If the oscillations about the lowest point be very small, we may obtain interesting results by an *approximate* solution.

Let θ be the angle between the axis of z and the radius through the particle, ψ the angle denoting the azimuth of the plane containing these two lines, ρ the distance of the particle from the axis. Let the projection be made horizontally with velocity V when $\theta = \alpha$, $\psi = 0$, $t = 0$ together.

$$\text{Then } z = a\cos\theta = a\left(1 - \frac{\theta^2}{2}\right), \text{ approximately,}$$

$$k = a\cos\alpha = a\left(1 - \frac{\alpha^2}{2}\right) \dots\dots\dots\dots\dots\dots$$

Also (2) gives at once,

$$\rho^2\frac{d\psi}{dt} = h = a^2\sin^2\alpha\,\frac{V}{a\sin\alpha} = aV\alpha, \text{ approximately} \dots\dots (5)$$

Hence by (4), $t = \int \frac{a^2\theta d\theta}{\sqrt{[a^2\theta^2\{V^2 - ga(\theta^2 - \alpha^2)\} - a^2V^2\alpha^2]}}$

$$= \sqrt{\frac{a}{g}} \int \frac{\theta d\theta}{\sqrt{\{(\alpha^2 - \theta^2)(\theta^2 - \beta^2)\}}} \quad \ldots\ldots \ldots (6).$$

if $\beta^2 = \frac{V^2}{ga}$ be not greater than α^2. If it be, the signs of the factors in the denominator must be changed.

Hence, the value of θ lies between α and β.

If therefore $\alpha = \beta$, or $V^2 = ga\alpha^2$, the particle will move in a *horizontal* circle, and therefore with uniform velocity. We have then what is called a *Conical Pendulum*, and it is easy to see from equation (5) that in this particular case we are not confined to an approximate solution; as the result just obtained is true whatever be the magnitude of the horizontal circle described, provided we take $V^2 = ga\frac{\sin^2\alpha}{\cos\alpha}$ when α is finite.

We may now put (6), supposing $\alpha > \beta$, into the form

$$t = \left(\frac{a}{g}\right)^{\frac{1}{2}} \int \frac{\theta d\theta}{\left\{\left(\frac{\alpha^2 - \beta^2}{2}\right)^2 - \left(\theta^2 - \frac{\alpha^2 + \beta^2}{2}\right)^2\right\}^{\frac{1}{2}}};$$

and if we introduce a new variable, ω, such that

$$\theta^2 = \frac{(\alpha^2 + \beta^2) + \omega(\alpha^2 - \beta^2)}{2},$$

we have

$$2t = \left(\frac{a}{g}\right)^{\frac{1}{2}} \int \frac{d\omega}{\sqrt{(1 - \omega^2)}};$$

$$\text{or } 2(t + C) = \left(\frac{a}{g}\right)^{\frac{1}{2}} \cos^{-1}\omega.$$

$$\text{But when } t = 0, \quad \theta = \alpha, \quad \omega = 1;$$

$$\text{hence } \omega = \cos 2\left(\frac{g}{a}\right)^{\frac{1}{2}} t;$$

$$\text{whence } \theta^2 = \frac{\alpha^2+\beta^2}{2} + \frac{\alpha^2-\beta^2}{2}\cos 2\left(\frac{g}{a}\right)^{\frac{1}{2}} t;$$

or, substituting for the cosine of the double arc,

$$\theta^2 = \alpha^2 \cos^2\left(\frac{g}{a}\right)^{\frac{1}{2}} t + \beta^2 \sin^2\left(\frac{g}{a}\right)^{\frac{1}{2}} t \text{ .... (7).}$$

The value of θ^2 is therefore periodic. For $t = 0,\ \pi\left(\frac{a}{g}\right)^{\frac{1}{2}}$, $2\pi\left(\frac{a}{g}\right)^{\frac{1}{2}}$, &c. we have $\theta = \alpha$; and for $t = \frac{\pi}{2}\left(\frac{a}{g}\right)^{\frac{1}{2}}, \frac{3\pi}{2}\left(\frac{a}{g}\right)^{\frac{1}{2}}$, &c $\theta = \beta$. Hence the period is $\pi\left(\frac{a}{g}\right)^{\frac{1}{2}}$.

197. To find the motion of the plane in which θ is measured, we return to the equation (5),

$$\rho^2 \frac{d\psi}{dt} = aV\alpha; \text{ which gives } d\psi = \frac{V\alpha\, dt}{a\theta^2} = \left(\frac{g}{a}\right)^{\frac{1}{2}} \frac{\alpha\beta}{\theta^2}\, dt,$$

or, by (7),

$$= \left(\frac{g}{a}\right)^{\frac{1}{2}} \alpha\beta \frac{dt}{\alpha^2\cos^2\left(\frac{g}{a}\right)^{\frac{1}{2}} t + \beta^2 \sin^2\left(\frac{g}{a}\right)^{\frac{1}{2}} t};$$

the integral of which (§ 143) is

$$\psi + C = \tan^{-1}\left\{\frac{\beta}{\alpha}\tan\left(\frac{g}{a}\right)^{\frac{1}{2}} t\right\}.$$

But $\psi = 0,\ t = 0$ together; this gives $C = 0$, and finally

$$\tan\psi = \frac{\beta}{\alpha}\tan\left(\frac{g}{a}\right)^{\frac{1}{2}} t \text{ ....(8).}$$

It is easy from this to deduce the following results, viz. that each quarter revolution of this plane is accomplished in the same time, and simultaneously with the change of θ in that plane from α to β, or β to α. Also that, whatever we take as

the initial position, the time of this plane's turning through two right angles is the same, namely, $\pi\left(\frac{a}{g}\right)^{\frac{1}{2}}$.

198. If we eliminate t between (7) and (8), we find

$$\theta^2 = \frac{\alpha^2\beta^2}{\alpha^2 \sin^2\psi + \beta^2 \cos^2\psi}.$$

This is of the same form as the polar equation to an ellipse about the center. The projection of the particle's path on a horizontal plane is therefore approximately an ellipse, its semiaxes being $a\alpha$, $a\beta$.

199. *To determine approximately the apsidal angle.*

At an apse z is of course a maximum or minimum, and therefore $\frac{dz}{dt} = 0$. This gives, by § 195 (4),

$$(a^2 - z^2)\{V^2 - 2g(k - z)\} - h^2 = 0 \quad \text{.........} \quad (1),$$

whose two positive roots are the alternate values of z at the apses. Since we have supposed the particle to have been projected horizontally, the point of projection is an apse; and therefore k is a root of this equation.

Substituting k for z, we get

$$h^2 = V^2(a^2 - k^2);$$

therefore (1) becomes after reduction

$$(k - z)\{(k + z)V^2 - 2g(a^2 - z^2)\} = 0 \quad \text{.........} \quad (2).$$

And, if l be the other positive root of (1) or (2), we have

$$V^2 = \frac{2g(a^2 - l^2)}{k + l},$$

$$\text{and} \quad h^2 = \frac{2g(a^2 - k^2)(a^2 - l^2)}{k + l}.$$

Also if $-\gamma$ be the third root of (1),

$$\gamma - l = \frac{V^2}{2g}, \text{ or by (2)}, = \frac{a^2 - l^2}{k+l}.$$

$$\text{Hence } a + \gamma = \frac{(a+k)(a+l)}{k+l}.$$

$$\text{Now } \frac{d\psi}{dt} = \frac{h}{a^2 - z^2}; \text{ and therefore}$$

$$\frac{d\psi}{dz} = \frac{h}{a^2 - z^2} \frac{a}{\sqrt{\{2g(z-k)(l-z)(\gamma+z)\}}}.$$

Hence the apsidal angle or the value of ψ from $z = k$ to $z = l$, is

$$\psi_0 = a \left\{\frac{(a^2 - k^2)(a^2 - l^2)}{k+l}\right\}^{\frac{1}{2}} \int_k^l \frac{dz}{(a^2 - z^2)\sqrt{\{(z-k)(l-z)(\gamma+z)\}}}.$$

To get rid of γ put $z = a - \varpi$, the integral becomes

$$\int_{a-l}^{a-k} \frac{d\varpi}{\varpi(2a-\varpi)\left[\{(a-k)-\varpi\}\{\varpi-(a-l)\}\left\{\frac{(a+k)(a+l)}{k+l} - \varpi\right\}\right]^{\frac{1}{2}}};$$

and, expanding in powers of ϖ those factors whose variation is small compared with themselves, we have finally

$$\psi_0 = \tfrac{1}{2}\sqrt{\{(a-k)(a-l)\}} \int_{a-l}^{a-k} \frac{d\varpi}{\varpi\sqrt{[\{(a-k)-\varpi\}\{\varpi-(a-l)\}]}}$$

$$\left[1 + \frac{\varpi}{2}\left\{\frac{1}{a} + \frac{k+l}{(a+k)(a+l)}\right\} + \&c.\right]$$

$$= \frac{\pi}{2}\left[1 + \tfrac{1}{4}\sqrt{\{(a-k)(a-l)\}}\left\{\frac{1}{a} + \frac{k+l}{(a+k)(a+l)}\right\} + \ldots\right].$$

The integration may easily be carried on farther, all the terms being evidently positive, but we have enough to shew that the apsidal angle is greater than $\frac{\pi}{2}$, and that therefore in

the approximate elliptic path considered in last article the apse continually progredes.

In the case of this orbit, if p and q be its semiaxes, we have, by the properties of the sphere,

$$\left.\begin{aligned} a-k &= \frac{p^2}{2a} \text{ nearly} \\ a-l &= \frac{q^2}{2a} \ldots\ldots\ldots \end{aligned}\right\};$$

and therefore the apsidal angle

$$= \frac{\pi}{2}\left(1 + \frac{3}{8}\frac{pq}{a^2} + \ldots\right),$$

and the rate of progression of the apse therefore varies as the area of the projected orbit nearly.

200. *To determine the nature of the small oscillations executed under the action of gravity, on a smooth surface, by a particle about a position of stable equilibrium.*

The tangent plane at the position of equilibrium must be horizontal, and the surface must evidently lie *above* it in order that the equilibrium may be stable.

If ρ, ρ_1 be the radii of curvature of the principal normal sections, and if the axes of x and y be tangents to these sections respectively, at the point of contact with the horizontal plane, we know by Analytical Geometry that the equation to the surface in the immediate neighbourhood of the origin is

$$2z - \frac{x^2}{\rho} - \frac{y^2}{\rho_1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equations of motion of the particle are, as in § 191,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= R\lambda \\ \frac{d^2y}{dt^2} &= R\mu \\ \frac{d^2z}{dt^2} &= R\nu - g \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

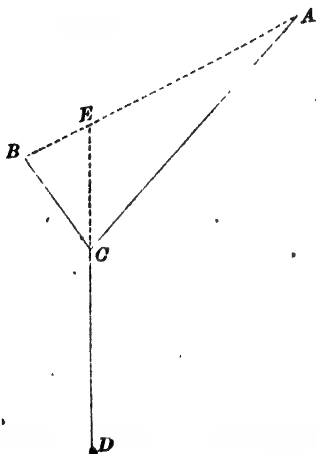
where λ, μ, ν are the direction-cosines of the normal to the surface at the point x, y, z. Since x and y are very small, z is of the second order of small quantities by (1) and may therefore be neglected, as may also $\frac{d^2z}{dt^2}$.

Hence $\lambda = -\frac{x}{\rho}$, $\mu = -\frac{y}{\rho_1}$, $\nu = 1$, approximately. Eliminating R from equations (2), we have

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= -\frac{g}{\rho}x\\ \frac{d^2y}{dt^2} &= -\frac{g}{\rho_1}y\end{aligned}\right\}\qquad(3),$$

which show (§ 173) that the motion consists of superposed simple pendulum oscillations in the principal planes, the lengths of the pendulums being the corresponding radii of curvature.

The annexed cut shows a very simple arrangement, due to Prof. Blackburn of Glasgow, by which this species of con-



straint may easily be produced. Three strings are knotted together at the point C, the other ends A and B of two of them

are attached to fixed points, and the third supports the heavy particle D. Suppose CE to be vertical, then the small oscillations of D will evidently be executed as if on a smooth surface whose principal planes of curvature at D are in, and perpendicular to, the plane of the paper. The radii of curvature in these planes are CD and DE respectively.

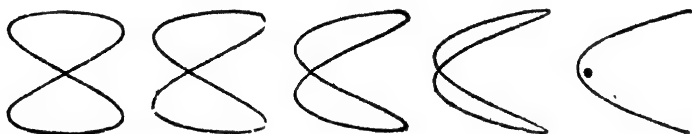
If we put $\frac{g}{\rho} = n^2$, and $\frac{g}{\rho_1} = n_1^2$, the integrals of (3) are

$$\left.\begin{aligned} x &= A \cos (nt + B), \\ y &= A_1 \cos (n_1 t + B_1). \end{aligned}\right\} \quad (4).$$

The curves corresponding to these equations are very interesting, but we cannot enter at length on the consideration of them. We may take, as a special case, that in which $DE = 4CD$; in which therefore

$$\left.\begin{aligned} x &= A \cos (nt + B), \\ y &= A_1 \cos (2nt + B_1). \end{aligned}\right\} \quad (5).$$

The circumstances of projection determine in each case the particular curve described—a few of the principal forms are sketched below, one of which is a portion of a parabola.



When n_1 is nearly, but not exactly, equal to $2n$, the curve described is always for a short time approximately one of the above figures, but its form slowly passes in succession from one member of the series to the next, completing the round when one pendulum has executed one more or less than twice as many complete oscillations as the other.

201. *To find the Brachistochrone for a particle constrained to move on a given smooth surface, gravity being the only impressed force.*

Let

$$F = 0 \text{ ....(1)}$$

be the equation to the given surface, z being the vertical axis.

$$\text{Then } \frac{ds}{dt} = \sqrt{\{2g(z - z_0)\}},$$

and therefore the time between the given points is

$$t_0 = \int_{z_0}^{z_1} \frac{\frac{ds}{dz}\,dz}{\sqrt{2g(z - z_0)}} \text{ ....(2).}$$

From the condition that t_0 is to be a minimum we obtain

$$\frac{d}{dz}\left\{\frac{\frac{dx}{ds}}{\sqrt{(z - z_0)}}\right\}\delta x + \frac{d}{dz}\left\{\frac{\frac{dy}{ds}}{\sqrt{(z - z_0)}}\right\}\delta y = 0.$$

But δx and δy are not independent, (1) gives us

$$\left(\frac{dF}{dx}\right)\delta x + \left(\frac{dF}{dy}\right)\delta y = 0.$$

Hence, eliminating, we obtain

$$\frac{\frac{d}{dz}\left\{\frac{dx}{ds}\frac{1}{\sqrt{(z - z_0)}}\right\}}{\left(\frac{dF}{dx}\right)} = \frac{\frac{d}{dz}\left\{\frac{dy}{ds}\frac{1}{\sqrt{(z - z_0)}}\right\}}{\left(\frac{dF}{dy}\right)},$$

which, by means of (1), may be reduced to a differential equation of the second order between two variables; the integral will therefore contain two arbitrary constants, which will enable us to make the curve pass through the two given points.

202. *A particle acted on by any forces, and resting on a smooth horizontal plane, is attached by an inextensible string*

to a point which moves in a given manner in that plane; to determine the motion of the particle.

Let x, y, $\bar{x}$, $\bar{y}$ be the co-ordinates, at time t, of the particle and point, a the length of the string, and R the force of constraint.

For the motion of the particle we have

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= X - R\,\frac{x-\bar{x}}{a} \\ \frac{d^2y}{dt^2} &= Y - R\,\frac{y-\bar{y}}{a}\end{aligned}\right\}\ldots\ldots\ldots\ldots(1),$$

with the condition $(x-\bar{x})^2 + (y-\bar{y})^2 = a^2$.

Now $\bar{x}$, $\bar{y}$ are given functions of t. Take from both sides of the equations in (1) the quantities $\frac{d^2\bar{x}}{dt^2}$, $\frac{d^2\bar{y}}{dt^2}$, respectively, and we have the equations of *relative* motion

$$\left.\begin{aligned}\frac{d^2(x-\bar{x})}{dt^2} &= X - R\,\frac{x-\bar{x}}{a} - \frac{d^2\bar{x}}{dt^2} \\ \frac{d^2(y-\bar{y})}{dt^2} &= Y - R\,\frac{y-\bar{y}}{a} - \frac{d^2\bar{y}}{dt^2}\end{aligned}\right\}\ldots\ldots(2).$$

These are precisely the equations we should have had if the point had been fixed, and in addition to the forces X, Y and R acting on the particle, we had applied, reversed in direction, the accelerations of the point's motion. It is evident that the same theorem will hold in three dimensions. The accelerations $\frac{d^2\bar{x}}{dt^2}$, $\frac{d^2\bar{y}}{dt^2}$ are known as functions of t, and therefore the equations of relative motion are completely determined. Compare § 24.

203. *Let there be no impressed forces, and suppose first that the point moves uniformly in a straight line.*

Here $\frac{d\bar{x}}{dt}$, $\frac{d\bar{y}}{dt}$ are constant, and therefore no terms are

introduced in the equations of motion. We have thus the case of § 26.

Again, *suppose the point's motion to be rectilinear, but uniformly accelerated.*

The relative motion will evidently be that of a simple pendulum from side to side of the point's line of motion. In certain cases, when the angular velocity exceeds a certain limit, we shall have the string occasionally untended; and this will give rise to an impact (Chap. X.) when it is again tended. While the string is untended the particle moves, of course, in a straight line.

204. *Suppose the point to move, with uniform angular velocity ω, in a circle whose radius is* r *and center origin.*

Here, supposing the point to start from the axis of x,

$$\bar{x} = r \cos \omega t, \quad \bar{y} = r \sin \omega t.$$

Hence the equations of motion are, since

$$\frac{d^2\bar{x}}{dt^2} = -\omega^2\bar{x}, \quad \frac{d^2\bar{y}}{dt^2} = -\omega^2\bar{y},$$

$$\left.\begin{aligned} \frac{d^2(x-\bar{x})}{dt^2} &= -\frac{R}{m}\frac{x-\bar{x}}{a} + \omega^2\bar{x}, \\ \frac{d^2(y-\bar{y})}{dt^2} &= -\frac{R}{m}\frac{y-\bar{y}}{a} + \omega^2\bar{y}, \\ (x-\bar{x})^2 + (y-\bar{y})^2 &= a^2. \end{aligned}\right\}$$

Whence $(x-\bar{x})\dfrac{d^2(y-\bar{y})}{dt^2} - (y-\bar{y})\dfrac{d^2(x-\bar{x})}{dt^2}$

$$= \omega^2\{(x-\bar{x})\bar{y} - (y-\bar{y})\bar{x}\};$$

or, in polar co-ordinates, for the relative motion,

$$\frac{d}{dt}\left(a^2\frac{d\theta}{dt}\right) = -\omega^2 ar \sin(\theta - \omega t),$$

$$\text{or} \quad \frac{d^2(\theta - \omega t)}{dt^2} = -\omega^2\frac{r}{a}\sin(\theta - \omega t).$$

Now $\theta - \omega t$ is the inclination of the string to the radius passing through the point; call it ϕ, and we have

$$\frac{d^2\phi}{dt^2} = -\omega^2 \frac{r}{a} \sin\phi,$$

which is the ordinary equation of motion of a simple pendulum whose length is $\frac{ga}{r\omega^2}$.

The particle therefore moves, with reference to the uniformly revolving radius of the circle described by the point, just as a simple pendulum with reference to the vertical.

205. *To determine the motion of a particle acted on by given forces, and constrained to move in a smooth tube, in the form of a given plane curve, of indefinitely small sectional area, which revolves in a given manner about an axis in its plane.*

Let the axis of revolution be that of z, and let the position of the particle at time t be given by its distance r from that axis, the plane of the tube at that instant making an angle θ with a fixed plane passing through the axis. By the conditions of the problem θ is a given function of t.

The sole effect of the tube will be to produce a force of constraint, which lies in the normal plane to the tube, and may therefore be resolved into two parts, one perpendicular to the plane of the tube, the other in that plane and in the principal normal to the tube.

Let the impressed forces be resolved into three, P along r, T perpendicular to the plane of the tube, and S parallel to the axis of z.

Let R, R' be the two resolved parts of the force of constraint.

The equations of motion will then be (by §§ 15, 64)

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = P + R\frac{dz}{ds} \quad\ldots\ldots\ldots\ldots\ldots(1),$$

$$r\frac{d^2\theta}{dt^2}+2\frac{dr}{dt}\frac{d\theta}{dt}=T-R' \quad\ldots\ldots\ldots\ldots\ldots\ (2),$$

$$\frac{d^2z}{dt^2}=S-R\frac{dr}{ds} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3),$$

where s is the arc of the revolving curve.

In addition to these we have the two equations

$$\theta=f(t) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (4),$$

which gives the position of the tube at any time, and

$$r=\phi(z) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (5),$$

the equation to the tube.

By means of (4) and (5) we may eliminate θ, r, and s from (1), (2), (3). Then eliminating R between (1) and (3), we obtain a differential equation between z and t, whose integral together with (4) completely determines the *position* of the particle at any instant.

R and R' may then be found from (1) or (3), and (2).

In general the angular velocity of the tube is given constant, or $\frac{d\theta}{dt}=\omega$, whence (4) becomes $\theta=\omega t$ if the plane from which θ is measured be that of the tube at the time $t=0$.

The simplest case we can take is the following.

206. *A particle moves in a smooth straight tube which revolves uniformly round a vertical axis to which it is perpendicular, to determine the motion.*

Here $z=$ constant, $\frac{d\theta}{dt}=$ constant $=\omega$, $P=0$, and we have from (1)

$$\frac{d^2r}{dt^2}-r\omega^2=0;$$

whence $r=A\epsilon^{\omega t}+B\epsilon^{-\omega t}$.

Suppose the motion to commence at time $t=0$ by the cutting of a string, length r_0, attaching the particle to the axis. The velocity of the particle at that instant along the tube would be zero. Hence at $t=0$

$$r=r_0=A+B,$$

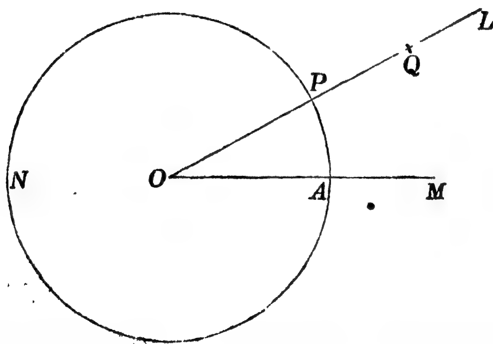
$$\frac{dr}{dt}=0=A-B;$$

$$\therefore\ A=B=\frac{r_0}{2};$$

$$\text{and}\ r=\frac{r_0}{2}\left(\epsilon^{\omega t}+\epsilon^{-\omega t}\right).$$

In the figure, let OM be the initial position of the tube, A that of the particle; OL, Q, the tube and particle at time t. Then $OA=r_0$, arc $AP=r_0\omega t$, $OQ=r$, and we have

$$OQ=\frac{OA}{2}\left(\epsilon^{\frac{\text{arc}\,AP}{OA}}+\epsilon^{-\frac{\text{arc}\,AP}{OA}}\right).$$



Whence we see that OQ and the arc AP are corresponding values of the ordinate and abscissa of a catenary whose parameter is OA.

Here, by (3), we have evidently $R=g$.

Also, by (2), $R' = -2\frac{r_0\omega}{2}(\epsilon^{\omega t} - \epsilon^{-\omega t})\,\omega$

$$= -\omega^2 r_0 (\epsilon^{\omega t} - \epsilon^{-\omega t}).$$

From this equation, combined with the value of r, we easily deduce

$$R' = 2\omega^2 \sqrt{(r^2 - r_0^2)},$$

and it is therefore proportional at any instant to the tangent drawn from Q to the circle APN.

207. *Suppose the tube to revolve uniformly in a vertical plane about a horizontal axis.*

We have from equation (1) of § 205

$$\frac{d^2r}{dt^2} - r\omega^2 = -g\cos\omega t,$$

if we conceive the tube to be vertical when $t = 0$. The integral of this equation is

$$r = A\epsilon^{\omega t} + B\epsilon^{-\omega t} - g\left\{\left(\frac{d}{dt}\right)^2 - \omega^2\right\}^{-1}\cos\omega t,$$

$$\text{or } r = A\epsilon^{\omega t} + B\epsilon^{-\omega t} + \frac{g}{2\omega^2}\cos\omega t;$$

$$\text{and if } r = r_0,\ \frac{dr}{dt} = 0, \text{ when } t = 0,$$

$$\text{we have } r_0 = A + B + \frac{g}{2\omega^2},$$

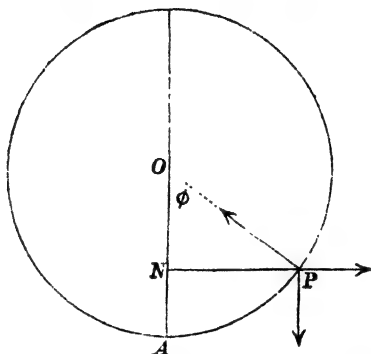
$$\text{and } 0 = A - B;$$

$$\text{or, } r = \left(\frac{r_0}{2} - \frac{g}{4\omega^2}\right)(\epsilon^{\omega t} + \epsilon^{-\omega t}) + \frac{g}{2\omega^2}\cos\omega t,$$

which completely determines the motion. R and R' may be found as before.

208. *Let the tube be in the form of a circle turning uniformly about a vertical diameter.*

Let AO be the axis, P the position of the particle at any time. Let $POA = \phi$ denote the particle's position. Then the forces acting on it along the element at P are the resolved



parts of gravity and of the centrifugal force (Chap. IX.) due to the distance NP from the axis. Hence as $AP = AO.\phi = a\phi$, we have

$$a\frac{d^2\phi}{dt^2} = a\omega^2 \sin\phi\cos\phi - g\sin\phi \;\ldots\ldots\ldots\ldots (1),$$

the first integral of which is evidently

$$\left(\frac{d\phi}{dt}\right)^2 = \omega^2\sin^2\phi + \frac{2g}{a}\cos\phi + C.$$

Suppose the particle to be projected from the lowest point with angular velocity ω_1; we have, from the last written equation,

$$\omega_1^2 = \frac{2g}{a} + C.$$

Hence $\left(\dfrac{d\phi}{dt}\right)^2 = \omega^2\left\{1 - \cos^2\phi - \dfrac{2g}{a\omega^2}(1-\cos\phi)\right\} + \omega_1^2.$

This will be zero when ϕ has a value determined from the equation

$$\cos^2\phi - \frac{2g}{a\omega^2}\cos\phi = 1 - \frac{2g}{a\omega^2} + \frac{\omega_1^2}{\omega^2};$$

$$\text{or } \cos\phi = \frac{g}{a\omega^2} \pm \sqrt{\left\{\left(1 - \frac{g}{a\omega^2}\right)^2 + \frac{\omega_1^2}{\omega^2}\right\}}.$$

Now, so long as $\frac{\omega_1^2}{\omega^2} > \frac{4g}{a\omega^2}$, or $\omega_1^2 > \frac{4g}{a}$, both values of $\cos\phi$ are numerically greater than 1, and therefore the motion is one of continuous revolution. If $\omega_1^2 = \frac{4g}{a}$, we have $\frac{d\phi}{dt} = 0$ for $\cos\phi = -1$, and therefore the particle just comes to rest at the highest point. We may notice that $a^2\omega_1^2$, or the square of the velocity of projection from the lowest point, is then equal to $2g.2a$, or the velocity is that due to the diameter.

Hence, if a particle be projected from the lowest point of a vertical circle with velocity due to the diameter, it will tend to reach the highest point and there remain at rest whether the circle be fixed or revolving with any angular velocity about the vertical diameter, another simple instance of conservation of energy. In this case the position of the particle at any instant can be determined.

If $\omega_1^2 < \frac{4g}{a}$, there is one possible value of $\cos\phi$, and therefore the particle will oscillate about the lowest point.

Suppose, again, the projection to be made from the extremity of the horizontal diameter. In this case our corrected equation becomes

$$\left(\frac{d\phi}{dt}\right)^2 = -\omega^2\cos^2\phi + \frac{2g}{a}\cos\phi + \omega_1^2;$$

and for positions of rest

$$\cos^2\phi - \frac{2g}{a\omega^2}\cos\phi = \frac{\omega_1^2}{\omega^2};$$

$$\text{or } \cos\phi = \frac{g}{a\omega^2} \pm \sqrt{\left(\frac{g^2}{a^2\omega^4} + \frac{\omega_1^2}{\omega_2^2}\right)}.$$

Both values will be numerically less than 1, if

$$\sqrt{\left(\frac{g^2}{a^2\omega^4}+\frac{\omega_1^2}{\omega^2}\right)}<1-\frac{g}{a\omega^2};$$

$$\text{or } \frac{\omega_1^2}{\omega^2}<1-\frac{2g}{a\omega^2},$$

and in this case the oscillations of the particle will be performed between points corresponding to these values of ϕ, and on the same side of the vertical diameter.

209. The position of equilibrium of the particle will be found by putting $\frac{d^2\phi}{dt^2}=0$. Hence, if ϕ' be the corresponding value of AOP,

$$\cos\phi'=\frac{g}{a\omega^2} \quad\text{.........................}\quad (2).$$

To find the time of a small oscillation about this position, let ψ be the angle of displacement, then by (1),

since $\phi=\phi'+\psi$, ψ being very small,

$$\frac{d^2\psi}{dt^2}=\omega^2\sin(\phi'+\psi)\left\{\cos(\phi'+\psi)-\frac{g}{a\omega^2}\right\}$$

$$=-\omega^2\sin^2\phi'.\psi \text{ nearly, by (2),}$$

$$=-\left(\omega^2-\frac{g^2}{a^2\omega^2}\right)\psi,$$

and therefore by § (85), the time required is

$$\frac{2\pi a\omega}{\sqrt{(a^2\omega^4-g^2)}} \quad\text{.........................}\quad (3).$$

That there may be a position of equilibrium other than the highest or lowest point, we must have by (2)

$$\omega>\sqrt{\left(\frac{g}{a}\right)},$$

and thus (3) shews that a small oscillation is always possible

when there is a position of equilibrium other than the highest or lowest point.

210. *Find the form of the tube in order that the particle projected with given velocity may preserve its velocity unchanged, gravity acting parallel to the axis.*

Resolving tangentially, and taking co-ordinates x, y in the plane of the curve, the axis of revolution being that of y, we have

$$\frac{d^2s}{dt^2} = x\omega^2\frac{dx}{ds} - g\frac{dy}{ds}.$$

$$\text{Hence, } \left(\frac{ds}{dt}\right)^2 = x^2\omega^2 - 2gy + C.$$

$$\text{But } \frac{ds}{dt} = \text{constant.}$$

Hence, $x^2 = \frac{2g}{\omega^2}(y+k)$, the equation to a parabola whose axis is vertical and vertex downwards. This result might easily have been foreseen, as the velocity can only be constant if the accelerating effect of the impressed forces along the curve be zero at every point; that is, if the resultant of gravity and centrifugal force lie in the normal. That this may be the case, we must have Centrifugal force : Gravity :: Ordinate : Subnormal. But the centrifugal force is proportional to the ordinate, hence the subnormal must be proportional to gravity, i.e. must be constant: a property peculiar to the parabola. This proposition has a singular application in Hydrostatics.

211. *A particle moves on a rough curve, under the action of given forces; to determine the motion.*

If μ' be the coefficient of kinetic friction, and

$$R = \sqrt{(R_1^2 + R_2^2)}$$

be the force of constraint as in § (186), the effect of friction will be a force $\mu'\sqrt{(R_1^2 + R_2^2)}$ acting in the tangent to the curve, and in the opposite direction to the particle's motion.

Equation (1) of § (186), will therefore become

$$\frac{d^2s}{dt^2} = S - \mu' \sqrt{(R_1^2 + R_2^2)},$$

the other two equations remaining the same.

If from the three we eliminate R_1 and R_2, we may by means of the equations to the curve eliminate x, y and z, and the final result, involving only s and t, suffices to determine the motion completely.

212. Ex. *A particle moves in a rough tube in the form of a plane curve, under the action of no forces; to determine the motion.*

Here $\frac{d^2s}{dt^2} = -\mu' R = -\frac{\mu' v^2}{\rho}$.

Now $v\frac{dv}{ds} = \frac{d^2s}{dt^2}$, hence

$$v\frac{dv}{ds} = -\mu' \frac{v^2}{\rho}:$$

$$\text{or } v = a\epsilon^{-\mu' \int \frac{ds}{\rho}}.$$

But, if ψ be the angle which the tangent at any point makes with a fixed line,

$$\frac{ds}{\rho} = d\psi.$$

Hence, $v = a\epsilon^{-\mu'\psi}$, where a is the velocity when $\psi = 0$.

It may be instructive to compare this result with that for the tension of a string stretched over a rough curve.

If the curve be one of double curvature, $\frac{ds}{\rho}$ is the angle between two successive tangents. If the surface of which the curve is the cuspidal edge be developed, and if ϕ represent the angle between the tangents corresponding to the initial and final positions of the particle,

$$v = a\epsilon^{-\mu'\phi}.$$

213. *A particle under the action of given forces moves on a given rough surface; to determine the motion.*

If R be the force of constraint due to the surface, the effect of friction is $\mu' R$ acting in the tangent to the path of the particle, and the equations of § 191 become

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= X + R\lambda - \mu' R \frac{dx}{ds} \\ \frac{d^2y}{dt^2} &= Y + R\mu - \mu' R \frac{dy}{ds} \\ \frac{d^2z}{dt^2} &= Z + R\nu - \mu' R \frac{dz}{ds} \end{aligned}\right\},$$

from which R must be eliminated. The two resulting equations contain x, y, z and t, and if the latter be eliminated, we have one equation in x, y, z which, with the equation to the surface, will completely determine the path. In general these equations are utterly intractable.

EXAMPLES.

(1) If a particle attached by a string to a point *just* make complete revolutions in a vertical plane, the tension of the string in the two positions when it is vertical is zero, and six times the weight of the particle, respectively.

(2) On a railway where the friction is $\frac{1}{240}$ of the load, shew that five times as much can be carried on the level as up an incline of 1 in 60 by the same power at the same rate.

(3) A pendulum which vibrates *seconds* at a place A, gains two beats per hour at a place B; compare the weights of any the same substance at the two places.

(4) From a point upon the surface of a smooth vertical circular hollow cylinder, and inside, a particle is projected in a direction making an angle a with the generating line through the point; find the velocity of projection that the

particle may rise to a given height (h) above the point, and the condition that the highest point may be vertically above the point of projection.

(5) A heavy particle rests on the arc of a smooth vertical circle at an angular distance of 30° from the lowest point, being repelled from one extremity of the horizontal diameter by a constant force; shew that, if slightly displaced along the arc, it will perform small oscillations in the time

$$2\pi\sqrt{\frac{a}{3\sqrt{3}g}}.$$

(6) A particle is constrained to move on a smooth curve under the action of a central force P tending to the pole, and the pressure on the curve varies always as the curvature, shew that

$$P \propto \frac{1}{p^3}\frac{dp}{dr}.$$

(7) A seconds pendulum when taken to the top of a mountain h miles high will lose $21.6h$ beats in a day nearly.

(8) AB is the diameter of a sphere of radius a; a centre of force at A attracts with a force ($\mu \times$ distance); from the extremity of a diameter perpendicular to AB a particle is projected along the inner surface with a velocity $(2\mu)^{\frac{1}{2}} a$: shew that the velocity of the particle at any point P is proportional to $\sin\theta$, and the pressure to $1-3\sin^2\theta$, where θ is the angle PAB.

(9) A chord AB of a circle is vertical and subtends at the centre an angle $2\cot^{-1}\mu$. Shew that the time down any chord AC drawn in the smaller of the two segments into which AB divides the circle is constant, AC being rough and μ the coefficient of friction.

(10) A particle under the action of no force is projected with velocity V in a rough tube in the form of an equiangular spiral at a distance a from the pole and towards the pole; shew that it will arrive at the pole in time

$$\frac{a}{V}\frac{1}{\mu \sin \alpha - \cos \alpha},$$

α being the angle of the spiral and μ ($> \cot \alpha$) the coefficient of friction.

(11) If a particle move on the surface of a smooth cone with its axis vertical and vertex downwards, and gravity be the only force acting, shew that the differential equation of the projection of its path on a horizontal plane is

$$\frac{d^2u}{d\theta^2} + u \sin^2 \alpha = \frac{g \sin \alpha \cos \alpha}{h^2u^2},$$

α being the semi-vertical angle of the cone.

(12) A particle is suspended from a fixed point by an inextensible string: find the velocity with which it must be projected when at the lowest point, so that its path after the string has ceased to be stretched may pass through the point of suspension.

(13) A particle is constrained to remain on the curve $r = a(1 - \cos \theta)$ and is repelled from the pole by a force $= \frac{\mu}{r^2}$: if its velocity at the apse be equal to $\left(\frac{\mu}{a}\right)^{\frac{1}{2}}$, shew that it will arrive at the initial line again in time $\pi \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}$.

(14) A particle slides down a catenary, whose plane is vertical and vertex upwards, the velocity at any point being that due to falling from the directrix; prove that the pressure at any point is inversely proportional to the distance of that point from the directrix.

(15) A particle projected with given velocity, moves under the action of gravity on a curve in a vertical plane; find the nature of the curve that the pressure on it may be the same throughout the motion.

(16) A particle is projected with given velocity from the

vertex of a cycloid whose axis is vertical and vertex uppermost, find where it will leave the curve, and the latus rectum of its future parabolic path.

√ (17) The axis major of an ellipse is vertical, shew that in order that a particle projected vertically upwards from the extremity of the axis minor along the concave side of the arc may pass through the center after leaving the curve, the velocity of projection must be

$$\left\{\frac{(8a^2+b^2)g}{3a\sqrt{3}}\right\}^{\frac{1}{2}},$$

a and b being the semiaxes of the ellipse.

(18) The Earth being supposed to be at rest, and to consist of concentric spherical strata with densities varying gradually from the centre to the surface, investigate the law of density according to which a particle let fall from the mouth of a diametral pit would perform oscillations exactly similar to those of a simple pendulum oscillating through 45° on each side of the vertical.

(19) Shew that if a particle falling from rest at a point in an inverted cycloid have its velocity suddenly annihilated when it has passed over half its vertical height above the lowest point, and be allowed to proceed always losing its velocity when half way down from the last position of no velocity, it will be at $\frac{1}{2^{2n}}$th of its original height above the vertex after n times the time it would have taken to fall to the vertex undisturbed.

— (20) Shew that a simple pendulum under the action of a central force varying as the distance only, will move as it does under the action of gravity.

— (21) The times of oscillation of a pendulum are observed at the Earth's surface, and also at a height h above the surface; from these data find the radius of the Earth supposed spherical.

(22) A pendulum oscillates in a small circular arc, and

is acted on in addition to gravity by a small horizontal force as the attraction of a mountain. Shew how to find the latter by observing the number of oscillations gained in a given time. Also find the direction in which the attraction must act so as not to alter the time of oscillation.

(23) (See 15). Determine the nature of the curve about which the string of a simple pendulum must wrap itself in order that its tension may be constant, and deduce the equation between the length of the arc, and the vertical ordinate

$$y = l - \frac{T}{g}(l-s) + C(l-s)^{\frac{2}{3}} - \frac{V^2}{2g},$$

where l is the length of the string, T the constant tension, and V the velocity of the bob when the string is vertical.

(24) A string wrapped round a regular polygon has a particle at the free end which just reaches an angle. There is in the center a repulsive force $\propto (D)$. If v_r be the velocity when r sides are unwrapped, shew that

$$v_r^2 = \frac{r(r+1)}{2} v_1^2,$$

the particle starting from rest.

(25) Find the curve cutting a series of ellipses with the same vertical axis and vertex, so that a particle descending each of them from rest at the point of section may press equally at the vertex; in the following cases,

(α) vertical axes $= a$.

(β) horizontal axes $= b$.

(γ) ellipses similar.

(26) Find the equation to a curve in a vertical plane, such that if a particle descend along it, the parts of the pressure due to the velocity, and to gravity, may have a given ratio.

If this be e, and the axis of y be vertical, then the differential equation is

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{C}{y}\right)^2 - 1,$$

where C is a constant.

(27) Find a curve such that a particle starting from rest will describe any arc in the same time as the chord, the force being central and $\propto$ distance.

Deduce (180) as a particular case.

(28) Also find the curve in (180) when the time down the chord is in a given ratio to that down the arc.

(29) Find the curve in which a particle acted on by gravity will revolve uniformly about a point in the same vertical plane.

(30) A particle acted on by a central repulsive force varying as the distance moves in a tube of the form of an epicycloid, the pole being at the center of force. Shew that the oscillations are isochronous.

(31) A particle is initially at rest at a point of the spiral $r = c\epsilon^{-m\theta}$, distant d from the pole. Shew that if the pole be a center of force whose attraction $= \frac{\mu}{D^2}$, the time of fall to it is

$$\frac{\pi}{2} \frac{d^{\frac{3}{2}}}{\sqrt{(2\mu)}} \sqrt{\left(1 + \frac{1}{m^2}\right)}.$$

(32) In the preceding problem find the pressure on the curve at any instant.

(33) A particle starts from rest at any point of the convex circumference of a vertical circle, shew that it will leave the circle after descending one-third of its original vertical height above the center.

(34) A particle under the influence of gravity is projected from one point in a horizontal direction towards another point, find the curve on which it must be constrained to move so as to approach uniformly to the latter point.

(35) Two particles are projected from the same point, in the same direction, and with the same velocity, but at different instants, in a smooth circular tube of small bore whose plane is vertical, to shew that the line joining them constantly touches another circle.

Let the tube be called the circle A, and the horizontal line, to a fall from which the velocity is due, L. Let m, m' be corresponding positions of the particles. Suppose that mm' passes into its next position by turning about O, these two lines will intercept two indefinitely small arcs at m and m', which (by a property of the circle) are in the ratio $mO : m'O$.

Let another circle B be described touching mm' in O, and such that L is the radical axis of A and B. Let a be the distance between their centers, mp, $m'p'$ perpendiculars on L. Let mp cut A again in q and B in r, s.

Then by Geometry,

$$mO^2 = rm.ms = pm\,(rm - qs) = 2a.pm = \frac{a}{g}\,(\text{velocity of } m)^2.$$

Similarly,

$$m'O^2 = 2a.p'm' = \frac{a}{g}\,(\text{velocity of } m')^2.$$

Hence the velocities of m and m' are as $mO : m'O$, and therefore by what we have shewn above about elementary arcs at m and m', the proximate position of mm' is also a tangent to B, which proves the proposition.

It is easily seen from this, that if *one* polygon of a given number of sides can be inscribed in one circle and circumscribed about another, an indefinite number can be drawn. For this we have only to suppose a number of particles moving in A with velocities due to a fall from L, if they form at any time the angular points of a polygon whose sides touch B, they will continue to do so throughout the motion. This however does not belong to our subject.

(36) A particle under the action of gravity is projected with given velocity from a point, find the curve on which it must be constrained to move so as to recede uniformly from the point of projection.

(37) A railway train travels due north with given velocity. Compare the horizontal pressure on the rails due to the Earth's rotation, with the weight of the train.

(38) A particle attached by a string to a point moves in a horizontal plane. A small ring passing round the string moves uniformly in a straight line from the point. Shew how to find the equation to the actual path, and shew that that relative to the ring has the equation

$$r\theta = C.$$

(39) A particle descends from rest under the action of gravity. Find the curve on which it must move in order that the ratio of the times of descending two vertical spaces whose ratio is given, may also be equal to a given quantity.

Verify the general result in the particular cases:

(α) Double the height in double the time.

(β) Four times the height in eight times the time.

(40) s is the arc, and y the vertical ordinate of a curve passing through the origin. If time through s : time through chord of s :: ks : chord, shew that

$$s = Ay^{\frac{k}{2k-1}}.$$

(41) Given an arc of a curve, find the position in which it must be fixed, that a particle starting from rest may describe it under the action of gravity in the least time. Apply the result to an arc of the cardioide measured from the cusp.

(42) A series of similar and similarly situated curves start from a point A in a vertical plane; to find the synchronous curve, or that which cuts off from each of the series a portion which would be described in a given time τ by a particle starting from rest at A.

Taking A as origin, and the axis of y vertically downwards, we have

$$\tau = \int_0^y \sqrt{\left\{\frac{1+\left(\frac{dx}{dy}\right)^2}{2gy}\right\}}\, dy \text{ .... (1).}$$

Now the common equation of the curves contains *one* arbitrary parameter a, which will therefore appear in (1) when x is eliminated. Supposing then that (1) can be integrated, if between the integral and the equation to the curves we eliminate a, the result will be the equation to the required series of synchronous curves, in which τ will appear as a variable parameter.

The equation to the given series of curves will of course be in the form $\frac{x}{a} = f\left(\frac{y}{a}\right)$, so that if $\frac{y}{a} = \omega$, $\frac{dx}{dy}$ in terms of ω will not involve a. It is thence easy to deduce the following values of x and y for the required curve, in terms of ω,

$$x = \frac{\tau^2 f(\omega)}{\left[\int_0^\omega \sqrt{\left\{\frac{1+\left(\frac{dx}{dy}\right)^2}{2g\omega}\right\}}\, d\omega\right]^2},$$

$$y = \frac{\tau^2 \omega}{\left[\int_0^\omega \sqrt{\left\{\frac{1+\left(\frac{dx}{dy}\right)^2}{2g\omega}\right\}}\, d\omega\right]^2},$$

from which, if (1) is not integrable, the required curve can be constructed by quadratures.

(43) If t be the time in a cycloid from the point whose abscissa is the radius of the generating circle, to any other point; τ the time down the chord of the generating circle corresponding to the same two points, shew that

$$2\tan^{-1}\tau\sqrt{\frac{g}{2}} = \tan^{-1}\left(\sqrt{2}\tan t\sqrt{\frac{g}{2}}\right).$$

(44) A particle moves in a circular groove radius a under the action of a center of force $\propto D^{-2}$ situated at a distance b from the center of the circle. It is projected from the nearest point with velocity V, shew that for a complete revolution

$$V^2 \nless \frac{4\mu b}{a^2 - b^2}.$$

(45) A particle acted on by gravity descends from rest at a given point, find the nature of the curve on which it must move that the pressure may be at any instant as the square of the vertical height fallen through.

(46) Find the tautochrone when the force is as the cube root of the distance from the axis of x, and parallel to that of y.

$$x^{\frac{4}{3}} + y^{\frac{4}{3}} = a^{\frac{4}{3}}.$$

(47) To find all the tautochrones when the force is central, and varies as the distance.

If S be the force resolved along the curve, we must have

$$\frac{dS}{ds} = -\frac{\pi^2}{4\tau^2} \text{ (§ 188).}$$

Now if ϕ be the angle between the radius vector and tangent,

$$S = -\mu r \cos\phi = -\mu r \frac{dr}{ds};$$

$$\therefore \frac{d\left(r\dfrac{dr}{ds}\right)}{ds} = \frac{\pi^2}{4\mu\tau^2};$$

$$\therefore r\frac{dr}{ds}\frac{d}{ds}\left(r\frac{dr}{ds}\right) = \frac{\pi^2}{4\mu\tau^2} r\frac{dr}{ds},$$

$$\text{or } \left(r\frac{dr}{ds}\right)^2 = \frac{\pi^2}{4\mu\tau^2} r^2 + C;$$

$$\text{that is, } r^2 - p^2 = \frac{\pi^2}{4\mu\tau^2} r^2 + C.$$

And if a be that perpendicular from the center on the tangent which meets the latter at its point of contact, the corresponding value of r is a also, and therefore

$$C = -\frac{\pi^2}{4\mu\tau^2} a^2;$$

$$\therefore p^2 = \left(1 - \frac{\pi^2}{4\mu\tau^2}\right) r^2 + \frac{\pi^2}{4\mu\tau^2} a^2 \ldots\ldots\ldots\ldots\ldots (1),$$

which is the differential equation to the required curves.

The curves will differ in species according to the value of $\frac{\pi^2}{4\mu\tau^2}$.

I. Let $\frac{\pi^2}{4\mu\tau^2} > 1$, (1) takes the form

$$p^2 = e^2 (c^2 - r^2),$$

which is at once recognized as the differential equation to the epicycloid traced by a point in the circumference of a small circle, which rolls on the inner surface of a large one whose center is the pole.

If we suppose the radius of the larger circle to be indefinitely increased, its circumference tends to become a straight line and the epicycloid to become a series of cycloids. The force in this case tends to be constant, and perpendicular to the bases of these cycloids, whence in the limit we have the result of § (173).

II. Let $\frac{\pi^2}{4\mu\tau^2} = 1$.

Here $p = a$ and the curve is a straight line. This is the case of § (89).

III. $\frac{\pi^2}{4\mu\tau^2} < 1$. In this, as in the other cases, we may find the equation to the curve by integrating (1) after substituting for p its value in terms of r and θ, but the equation we thus obtain is very complicated. This curve is found to be a spiral with two symmetrical branches extending to an infinite distance, and approximating to spirals which make angles $= \cos^{-1} e$ with the radius vector.

IV. Let the origin be on the curve, and be the point to which the time of descent is measured. Then $a = 0$.

If $\frac{\pi^2}{4\mu\tau^2} > 1$, the equation is impossible.

If it $= 1$, we have a straight line through the origin.

If it < 1, we have the Logarithmic Spiral.

V. If the force be repulsive, we require only to change the sign of μ. In this case we have an epicycloid traced by a point of a circle which rolls on the outside of another whose center is origin. From this again we may deduce the tautochronism of the common cycloid for gravity.

(48) A particle moving on the interior surface of a vertical circular cylinder is projected with a given velocity, and goes round n times before it falls to the level of the point of projection. Determine the direction of projection.

(49) Shew that a particle moving under the action of gravity on a smooth helix whose axis is vertical, makes the first revolution from rest in the time

$$\sqrt{\frac{8\pi a}{g \sin 2\alpha}}.$$

(50) A groove is cut on a right cone of height h, at an angle β with the generating line. Shew that the time of reaching the base, from a vertical height h_1 below the vertex, by a particle sliding in the groove is

$$\frac{\sqrt{h} - \sqrt{h_1}}{\sqrt{(8g)} \cos \alpha \cos \beta},$$

where α is the semivertical angle.

(51) Find the curve on the surface of a vertical cylinder down which if a particle slide, the force of constraint will be constant.

(52) A particle moves on a smooth ellipsoid, under the action of a force $\propto (D)$ in the center. Given the velocity and direction with which it passes the extremity of an axis, find the pressure.

(53) A smooth tube of indefinitely small bore revolves in a horizontal plane. A particle attached to the axis by an elastic string moves in the tube. Determine the conditions that the motion be oscillatory.

(54) A circular tube of indefinitely small bore revolves with uniform angular velocity ω about a vertical diameter, and a particle in it is projected from the lowest point with velocity due to the diameter. Determine the motion, and shew that it is at its greatest distance from the axis after a time

$$\left(\frac{g}{a}+\omega^2\right)^{-\frac{1}{2}}\log_e\sqrt{\frac{\sqrt{\left(\frac{2g}{a}+\omega^2\right)}+\sqrt{\left(\frac{g}{a}+\omega^2\right)}}{\sqrt{\left(\frac{2g}{a}+\omega^2\right)}-\sqrt{\left(\frac{g}{a}+\omega^2\right)}}};$$

where a is the radius of the tube.

(55) The axis of a rough helix of radius a is vertical, and the curve makes an angle β with the horizon; a ring slides on it with initial velocity

$$\frac{\sqrt{(ga)}\,(\sin^2\beta-\mu^2\cos^2\beta)^{\frac{1}{2}}}{\sqrt{\mu}\cos\beta},$$

determine the motion.

— (56) A heavy particle attached to a point by a string whose unstretched length is a, lies on a rough horizontal plane and is projected perpendicular to the string with velocity v. If it comes to rest at a distance a from the point, after describing a distance s, $v^2=2\mu gs$.

(57) A particle descends a rough circular tube from the extremity of the horizontal diameter. If it stops at the lowest point, shew that

$$3\mu\epsilon^{-\mu\pi}+2\mu^2=1.$$

(58) Shew that the result of § (193) is true if the surface be rough. If a particle be projected with velocity V along the inner surface of a rough sphere, determine the motion, and shew that it will return to the point of projection in the time

$$\frac{r}{\mu' V}(\epsilon^{2\mu'\pi}-1),$$

where r is the radius of the sphere.

(59) If the only impressed force be a central one $= \frac{\mu}{D^n}$, and the velocity be that from infinity, shew that the equation to the brachistochrone is

$$r^{\frac{n+1}{2}} \cos \frac{n+1}{2} (\theta + \beta) = a^{\frac{n+1}{2}}.$$

(60) A material particle P, attached by a slender cord of given length a, to a point S in a fixed axis SA, is attracted by a constant force g in a direction always parallel to a line SB, which is inclined at a given angle to the axis SA, and revolves about it with a given angular velocity ω: shew that if $V =$ the velocity of P, $\omega' =$ the angular velocity of the plane PSA about SA, $\phi = \angle PSB$, $\theta = \angle PSA$,

$$V^2 = 2ga \cos \phi + 2a^2 \omega\omega' \sin^2 \theta + \text{const.}$$

Shew also that the dynamical conditions of this Problem are the same as those of a ball-pendulum acted upon by gravity, when the Earth's rotation is taken into account.

(61) A small smooth ring slides along a rod which moves with uniform angular velocity and so as always to be in contact with a given circle: determine the motion of the ring relatively to the rod.

(62) A ring slides on a smooth elliptic wire which moves in its own plane with uniform angular velocity about its center. Determine the motion; and find the time of a small oscillation about the position of equilibrium where this is possible.

(63) A particle is attached by a rod without mass to the extremity of another rod, n times as long, which revolves in a given manner about the other extremity, the whole motion taking place in a horizontal plane. If θ be the inclination of the rods, ω the angular velocity of the 2nd rod at the time t, prove that

$$\frac{d^2\theta}{dt^2} + \frac{d\omega}{dt} + n \left(\frac{d\omega}{dt} \cos \theta + \omega^2 \sin \theta \right) = 0.$$

(64) If a particle slide along a smooth curve, which turns with uniform angular velocity ω about a fixed point O, then the velocity of the particle relatively to the moving curve is given by the equation

$$v^2 = c^2 + \omega^2 r^2,$$

where r is the distance of the particle from the point O; and the pressure on the curve will be given by the formula

$$R = \frac{v^2}{\rho} + \omega^2 p + 2\omega v,$$

where p is the perpendicular from O on the tangent.

(65) A heavy particle is attached to a smooth string which passes over a rough circular arc in a vertical plane; the particle initially at the extremity of a horizontal diameter is drawn up with uniform acceleration $\frac{g}{\pi}$: shew that the whole labouring force (i. e. work, see § 55) expended in drawing it to the vertex of the circle is

$$Wa\left(\frac{3}{2} + \mu - \mu\frac{\pi}{4}\right),$$

where W is the weight of the particle, a the radius of the circle, and μ the coefficient of friction.

(66) A heavy particle is attached by a fine string to the apex of a right vertical cone whose semivertical angle is β, and is projected from a position of rest on the cone with an initial angular velocity ω (about its axis) which is less than Ω, the least angular velocity which would make the particle leave the cone. If the coefficient of friction between the particle and cone be μ, find the position of the particle and the tension of the string at a given instant; and shew that it will come to rest after a time

$$\frac{1}{2\mu\Omega\cos\beta}\log\frac{\Omega+\omega}{\Omega-\omega}.$$

(67) Determine (approximately) the motion of the bob of a simple pendulum; when the point of suspension describes uniformly, and with small velocity, a horizontal circle.

(68) If a curve revolve uniformly about a vertical axis and the only extraneous force be gravity, prove that the time of an oscillation of a particle sliding on the curve about its position of rest is

$$\frac{2\pi}{\omega}\sqrt{\frac{\rho \sin \alpha}{r - \rho \sin \alpha \cos^2 \alpha}},$$

ρ being the radius of curvature at the point of equilibrium, α the angle made by the normal at that point with the vertical, r the distance of the point from the axis of revolution, and ω the angular velocity of the curve.

CHAPTER VIII.

MOTION IN A RESISTING MEDIUM.

214. WHEN a body moves in a fluid, whether liquid or gaseous, it must, in displacing the particles of the medium and in rubbing against them, lose part of its own velocity. The resistance of a fluid to a body moving in it is therefore of the nature of a retarding force; but, in consequence of the great difficulty of making accurate experiments on the subject, the laws of the resistance of fluids have not yet been satisfactorily ascertained.

For a velocity neither very great nor very small, the general approximate law seems to be, that the resistance to a plane surface, moving with its plane at right angles to the line of motion, is proportional to the extent of the surface, the density of the resisting medium, and the square of the velocity taken conjointly. We are, however, only treating of the motion of a particle, in which the extent of surface has no place in our consideration, and will assume that the resistance depends entirely on the density of the medium and the velocity of the particle; illustrating, by supposing different laws, the method of procedure in all such cases.

215. *A particle acted on by no forces is projected in a resisting medium of uniform density, of which the resistance varies as the* n^{th} *power of the velocity; to determine the motion.*

The motion will evidently be rectilinear. Let x be the distance of the particle from a fixed point in the line of motion at the time t, v its velocity at that time. The force due to the resistance may be represented by kv^n, k being a constant, and the equation of motion is

$$\frac{d^2x}{dt^2} = -kv^n,$$

R

$$\text{or } \frac{dv}{dt} = -kv^n \quad \dots\dots\dots\dots (1).$$

Putting it in the form

$$k\frac{dt}{dv} = -v^{-n},$$

we have, by integration,

$$kt = C + \frac{v^{1-n}}{n-1}.$$

Suppose the particle to be projected from the origin, with velocity V, then when

$$t = 0,\ v = V;\ \text{and}\ C = -\frac{V^{1-n}}{n-1}.$$

$$\text{Hence}\ (n-1)\,kt = v^{1-n} - V^{1-n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

the relation between v and t. It shews that v can never be zero if $n > 1$, but if $n < 1$ the velocity will become zero when $t = \frac{V^{1-n}}{(1-n)\,k}$. After this the particle will evidently remain at rest.

To find the distance of the particle from the origin at any time, we have from (2)

$$v \text{ or } \frac{dx}{dt} = \{V^{1-n} - (1-n)\,kt\}^{\frac{1}{1-n}}.$$

$$\text{Hence } x = -\frac{1}{(2-n)\,k}\{V^{1-n} - (1-n)\,kt\}^{\frac{2-n}{1-n}} + \frac{V^{2-n}}{(2-n)\,k};$$

x and t being supposed to vanish together. When $n < 1$, the distance to which the particle will go, or its distance from the origin at the time $\frac{V^{1-n}}{(1-n)\,k}$, is

$$\frac{V^{2-n}}{(2-n)\,k}.$$

216. There is one case in which the above solution fails, namely when $n = 1$, or the resistance varies as the velocity. In this case, by (1),

$$\frac{dv}{dt} = -kv, \text{ and } k\frac{dt}{dv} = -\frac{1}{v},$$

$$\text{from which } kt = C - \log v.$$

When $t = 0$, $v = V$; and $C = \log V$.

$$\text{Hence } kt = \log\frac{V}{v} \text{.....................} (3);$$

$$\text{and therefore } v \text{ or } \frac{dx}{dt} = V\epsilon^{-kt}.$$

$$\text{Integrating,} \quad x = \frac{V}{k}(1 - \epsilon^{-kt}) \text{.....................} (4),$$

the constant being determined as before that x and t may vanish together.

Equations (3) and (4) determine the velocity and the position of the particle at any instant. They shew that the velocity continually diminishes without ever actually becoming zero, but that the space passed over by the particle can never be greater than a certain quantity, for when

$$t = \infty, \quad x = \frac{V}{k}.$$

217. *A particle, acted on by a constant force in its line of motion, moves in a resisting medium of uniform density, of which the resistance varies as the square of the velocity; to determine the motion.*

Suppose the particle projected from the origin with the velocity V, and let v be its velocity at any time t, x its distance from the origin at that time, and f the constant acceleration due to the force.

Assume K to be the velocity with which the particle would have to be animated that the resisting force might be equal to f, then the retarding force at any time may be represented by $f\frac{v^2}{K^2}$.

I. Let f act so as to diminish x; then the equation of motion is

$$\frac{d^2x}{dt^2} = -f - f\frac{v^2}{K^2},$$

which, since

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

may be written either

$$\frac{dv}{dt} = -\frac{f}{K^2}(K^2 + v^2)$$

$$\text{or } v\frac{dv}{dx} = -\frac{f}{K^2}(K^2 + v^2).$$

These again may be changed into

$$\frac{dt}{dv} = -\frac{K^2}{f}\frac{1}{K^2 + v^2},$$

$$\frac{dx}{d(v^2)} = -\frac{K^2}{2f}\frac{1}{K^2 + v^2}.$$

Integrating, and determining the constants so that when

$$x = 0, \ t = 0, \ v = V,$$

we obtain

$$\frac{ft}{K} = \tan^{-1}\frac{V}{K} - \tan^{-1}\frac{v}{K} = \tan^{-1}\frac{K(V - v)}{K^2 + Vv},$$

$$\frac{2fx}{K^2} = \log\frac{K^2 + V^2}{K^2 + v^2}.$$

Let T be the time at which the velocity becomes zero, and h the corresponding value of x, then

$$T = \frac{K}{f}\tan^{-1}\frac{V}{K}, \text{ and } h = \frac{K^2}{2f}\log\left(1 + \frac{V^2}{K^2}\right).$$

After this the particle begins to return, the force of resistance therefore tends to increase x, and the equation of motion is

$$\frac{d^2x}{dt^2} = -f + f\frac{v^2}{K^2},$$

which, as before, may be written either

$$-\frac{dv}{dt} = -\frac{f}{K^2}(K^2 - v^2),$$

$$\text{or } v\frac{dv}{dx} = -\frac{f}{K^2}(K^2 - v^2).$$

These may be changed into

$$\frac{dt}{dv} = \frac{K^2}{f}\frac{1}{K^2 - v^2},$$

$$\frac{dx}{d(v^2)} = -\frac{K^2}{2f}\frac{1}{K^2 - v^2}.$$

Integrating, and determining the constants so that when

$$v = 0, \; x = h, \; t = T,$$

we obtain

$$\frac{2f}{K}(t - T) = \log\frac{K + v}{K - v},$$

$$\frac{2f}{K^2}(h - x) = \log\frac{K^2}{K^2 - v^2}.$$

Let U be the velocity with which the particle will return to the point of projection; then, putting $x = 0$ in the latter equation, we obtain

$$\frac{U^2}{K^2} = 1 - \epsilon^{-\frac{2fh}{K^2}};$$

or, substituting for h its value,

$$\frac{U^2}{K^2} = \frac{\dfrac{V^2}{K^2}}{1 + \dfrac{V^2}{K^2}},$$

whence

$$\frac{1}{U^2} - \frac{1}{V^2} = \frac{1}{K^2}.$$

This shews, as we might expect, that the particle returns to the point of projection with diminished velocity.

II. Let f act so as to increase x. Then

$$\frac{d^2x}{dt^2} = f - f\frac{v^2}{K^2},$$

from which we derive, as before,

$$\frac{dt}{dv} = \frac{K^2}{f}\,\frac{1}{K^2 - v^2},$$

$$\frac{dx}{d(v^2)} = \frac{K^2}{2f}\,\frac{1}{K^2 - v^2}.$$

Integrating, and determining the constants so that when

$$t = 0,\ x = 0,\ v = V,$$

we obtain

$$t = \frac{K}{2f}\log\frac{(K+v)(K-V)}{(K-v)(K+V)},$$

$$x = \frac{K^2}{2f}\log\frac{K^2 - V^2}{K^2 - v^2}.$$

From the latter equation we obtain

$$v^2 = K^2 - (K^2 - V^2)\,\epsilon^{-\frac{2fx}{K^2}}.$$

This equation shews that when x becomes very large, v approaches to K, which is its limiting value. If the velocity of projection be less than K, v will continually approach to K, and never exceed it, and if the velocity be greater than K, v will constantly diminish towards K, and never become less.

218. The results of the last Proposition are applicable to bodies projected in a resisting medium vertically upwards or

downwards under the action of gravity; for the acceleration due to gravity may still be considered constant, although not the same as for a particle in vacuo. The actual moving force is in fact the difference of the weights of the body and the fluid displaced, so that if α be the ratio of the specific gravity of the fluid displaced to that of the body, the moving force

$$= W(1-\alpha) = Mg\,(1-\alpha),$$

where W and M are the weight and mass of the body, and therefore the acceleration caused by it $= g\,(1-\alpha)$. By substituting this for f in the results of § 217, we may obtain formulæ for the motion of bodies in a vertical direction under the action of gravity. Hailstones and raindrops afford a good illustration of the *Terminal Velocity* indicated by the result of II.

219. *To find the equations of motion, in a resisting medium, of a particle acted on by any forces.*

Let x, y, z be the co-ordinates of the particle relative to an assumed system of rectangular axes, at the time t, and let X, Y, Z be the component accelerations, parallel to the axes, due to the forces acting on the particle. Then denoting by R the acceleration due to the resistance, which lies in the tangent to the path described, and in a direction opposed to the motion, we have

$$\frac{d^2x}{dt^2} = X - R\frac{dx}{ds},$$

$$\frac{d^2y}{dt^2} = Y - R\frac{dy}{ds},$$

$$\frac{d^2z}{dt^2} = Z - R\frac{dz}{ds}.$$

These are the general equations of motion. In any particular case R will be given as a function of the density of the medium and the velocity of the particle, and particular methods will be necessary for obtaining the path of the particle and its position at any time. The method of procedure will be illustrated in what follows.

220. *A particle acted on by a constant force parallel to a fixed line is projected from a given point in a given direction with a given velocity, and moves in a uniform medium whose resistance varies as the square of the velocity; to determine the motion.*

This is, approximately, the case of a projectile disturbed by the resistance of the air; and its solution is, to a certain extent, useful in gunnery.

Take the given point as origin, the axis of x perpendicular, and that of y parallel, to the given line, so that the plane of xy may contain the direction of projection. Let f be the constant acceleration due to the force; acting, we will suppose, to diminish y; then the accelerating effect of the resistance may be represented by $k\left(\frac{ds}{dt}\right)^2$ where k is a constant. Hence the equations of motion are

$$\frac{d^2x}{dt^2} = -k\left(\frac{ds}{dt}\right)^2\frac{dx}{ds} \quad\text{.........................}(1),$$

$$\frac{d^2y}{dt^2} = -f - k\left(\frac{ds}{dt}\right)^2\frac{dy}{ds} \quad\text{....................}(2).$$

The former may be written

$$\frac{d^2x}{dt^2} = -k\frac{ds}{dt}\frac{dx}{dt};$$

and, therefore, dividing by $\frac{dx}{dt}$ and integrating,

$$\log\frac{dx}{dt} = C - ks.$$

Suppose u to be the component of the initial velocity parallel to x: then, when $s = 0$

$$\frac{dx}{dt} = u; \text{ whence } C = \log u, \text{ and}$$

$$\frac{dx}{dt} = ue^{-ks}.$$

Hence $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = u\epsilon^{-ks}\frac{dy}{dx}$;

therefore $\frac{d^2y}{dt^2} = u\epsilon^{-ks}\frac{d^2y}{dx^2}\frac{dx}{dt} - ku\epsilon^{-ks}\frac{ds}{dt}\frac{dy}{dx}$

$$= u^2\epsilon^{-2ks}\frac{d^2y}{dx^2} - ku\epsilon^{-ks}\frac{ds}{dt}\frac{dy}{dx}.$$

But by (2),

$$\frac{d^2y}{dt^2} = -f - k\frac{ds}{dt}\frac{dy}{dt}$$

$$= -f - ku\epsilon^{-ks}\frac{ds}{dt}\frac{dy}{dx}.$$

Comparing these we obtain

$$\frac{d^2y}{dx^2} + \frac{f}{u^2}\epsilon^{2ks} = 0;$$

or, putting $p = \frac{dy}{dx}$,

$$\frac{dp}{dx} + \frac{f}{u^2}\epsilon^{2ks} = 0 \qquad (3),$$

the differential equation to the path.

It may be made integrable once by multiplying the first term by

$$\frac{\sqrt{(1+p^2)}}{\frac{ds}{dx}}, \text{ which is unity.}$$

This gives $\frac{dp}{ds}\sqrt{(1+p^2)} + \frac{f}{u^2}\epsilon^{2ks} = 0.$

Integrating, and determining the constant so that when $s = 0$, $p = \tan\alpha$, α being the angle which the direction of projection makes with the axis of x, we obtain

$$p\sqrt{(1+p^2)} + \log\{p + \sqrt{(1+p^2)}\} + \frac{f}{ku^2}\epsilon^{2ks}$$

$$= \tan\alpha \sec\alpha + \log(\tan\alpha + \sec\alpha) + \frac{f}{ku^2} \dots\dots\dots (4),$$

the intrinsic equation to the path. This equation cannot be integrated farther.

If we make $k = 0$ in equations (1), (2), we have the equations which belong to the trajectory in a non-resisting medium, the original velocity and direction of projection being the same as in this problem. Hence if S, s be arcs of the trajectories in a non-resisting, and a resisting, medium measured from the point of projection to any two points at which the tangents are parallel,

$$\frac{dp}{ds}\sqrt{(1+p^2)} = -\frac{f}{u^2}\frac{dS}{ds} = -\frac{f}{u^2}\epsilon^{2ks}.$$

$$\text{Hence } \frac{dS}{ds} = \epsilon^{2ks};$$

and therefore $2kS = \epsilon^{2ks} - 1$,

since we suppose S and s to commence together.

$$\text{Hence } 2ks = \log(1 + 2kS) \dots\dots\dots\dots\dots\dots (5).$$

221. From equation (3) it appears that, as s becomes more and more nearly equal to $+\infty$, $\frac{dx}{dp}$ becomes more and more nearly zero, and therefore x becomes more and more nearly equal to a constant. Hence the curve on the positive side of the origin tends continually to coincide with a straight line parallel to the axis of y, at a finite distance, which is therefore an asymptote.

Again, as s becomes more and more nearly equal to $-\infty$, $\frac{dp}{dx}$ becomes more and more equal to zero, or p tends to become constant. The curve therefore on the negative side of the origin tends to become parallel to a certain straight line. It appears also from equation (5) that when $S = -\frac{1}{2k}$, s be-

comes $-\infty$, and therefore the curve tends to become parallel to the tangent at a point of the common parabola at a distance $-\frac{1}{2k}$ along the curve from the origin. Or, if θ be the angle which this straight line makes with the axis of x, we have by putting $s=-\infty$ in equation (4),

$$\tan\theta\sec\theta-\tan\alpha\sec\alpha+\log\frac{\tan\theta+\sec\theta}{\tan\alpha+\sec\alpha}=\frac{f}{ku^2}.$$

To shew that the curve has an asymptote parallel to this line, we must prove that $\bar{x}$, the distance of the intersection of the tangent with the axis, from the origin, is always finite.

$$\text{Now } \bar{x}=x-y\frac{dx}{dy};$$

$$\text{which gives } \frac{d\bar{x}}{dx}=-y\frac{d}{dx}\frac{dx}{dy}=\frac{y}{p^2}\frac{dp}{dx}.$$

Also, if the ultimate value of p be called n, we shall have ultimately,

$$y=nx,\quad s=x\sqrt{(1+n^2)},$$

and, by (3),

$$\frac{dp}{dx}=-\frac{f\epsilon^{2ks}}{u^2};$$

$$\therefore\ \frac{d\bar{x}}{dx}=-\frac{fx}{nu^2}\epsilon^{2kx\sqrt{(1+n^2)}};$$

$$\therefore\ \bar{x}=-\int_0^x\frac{fx}{nu^2}\epsilon^{2kx\sqrt{(1+n^2)}}dx,$$

which, by integration, will be found to be finite when x is infinite and negative.

Hence the curve is not similar on opposite sides of the vertex. The particle rises more obliquely and descends more vertically than it would do in a vacuum.

222. The projectile will have reached the highest point when $\frac{dy}{dx}=0$. This gives, for the length of the path between

the origin and the highest point, by equation (4), the expression

$$\frac{1}{2k}\log\left[1+\frac{ku^2}{f}\{\tan\alpha\sec\alpha+\log(\tan\alpha+\sec\alpha)\}\right],$$

and, for the velocity there, the value

$$u\left[1+\frac{ku^2}{f}\{\tan\alpha\sec\alpha+\log(\tan\alpha+\sec\alpha)\}\right]^{-\frac{1}{2}}.$$

223. The above results will, as in § 218, be made applicable to the motion of a body projected in the air under the action of gravity, by writing for f the value of g, corrected for buoyancy. The most important application of the problem is to the practice of rifled arms, in which case p is always small, and an approximate equation to the path may be found.

For we have

$$\frac{dp}{dx}+\frac{f}{u^2}\epsilon^{2ks}=0.$$

Multiplying this by $\frac{ds}{dx}=\sqrt{(1+p^2)}=1$, (neglecting higher powers of p than the first), we have

$$\frac{dp}{dx}+\frac{f}{u^2}\epsilon^{2ks}\frac{ds}{dx}=0.$$

Integrating, and observing that when $s=0$, $p=\tan\alpha$,

$$p+\frac{f}{2ku^2}\epsilon^{2ks}=\tan\alpha+\frac{f}{2ku^2};$$

$$\text{whence } \epsilon^{2ks}=1+\frac{2ku^2}{f}(\tan\alpha-p).$$

Substituting in equation (3), we obtain

$$\frac{dp}{dx}+\frac{f}{u^2}+2k(\tan\alpha-p)=0,$$

$$\text{or } \frac{dp}{dx}-2kp=-\left(\frac{f}{u^2}+2k\tan\alpha\right).$$

Multiply by ϵ^{-2kx} and integrate, determining the constant as before; and we have

$$p\epsilon^{-2kx} = -\frac{f}{2ku^2} + \left(\frac{f}{2ku^2} + \tan\alpha\right)\epsilon^{-2kx};$$

$$\text{or } p = \frac{dy}{dx} = -\frac{f}{2ku^2}\epsilon^{2kx} + \left(\frac{f}{2ku^2} + \tan\alpha\right).$$

Integrating, and observing that when $x = 0$, $y = 0$, we have finally,

$$y = x\left(\tan\alpha + \frac{f}{2ku^2}\right) - \frac{f}{4k^2u^2}(\epsilon^{2kx} - 1),$$

the approximate equation to the required path.

224. *A particle, moving in a resisting medium, is acted on by a force whose direction is constantly parallel to a fixed line; to find the resistance that a given curve may be described.*

Taking the fixed line as the axis of y, and denoting the force at any point by Y and the resistance by R, the equations of motion will be

$$\frac{d^2x}{dt^2} = -R\frac{dx}{ds} \quad\text{..............................}\quad (1),$$

$$\frac{d^2y}{dt^2} = Y - R\frac{dy}{ds} \quad\text{..............................}\quad (2).$$

Eliminating R,

$$\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2} = Y\frac{dx}{dt} \quad\text{....................}\quad (3).$$

$$\text{Now } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}};$$

$$\text{hence } \frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3},$$

$$=\frac{Y}{\left(\frac{dx}{dt}\right)^2};$$

$$\text{or}\quad \left(\frac{dx}{dt}\right)^2=\frac{Y}{\frac{d^2y}{dx^2}}\quad\text{.......................(4).}$$

Differentiating with respect to t,

$$2\frac{dx}{dt}\frac{d^2x}{dt^2}=\frac{d}{dx}\left(\frac{Y}{\frac{d^2y}{dx^2}}\right)\frac{dx}{dt};$$

$$\therefore\ \frac{d^2x}{dt^2}=\frac{1}{2}\frac{d}{dx}\left(\frac{Y}{\frac{d^2y}{dx^2}}\right);$$

and therefore by equation (1)

$$R=-\frac{1}{2}\frac{ds}{dx}\frac{d}{dx}\left(\frac{Y}{\frac{d^2y}{dx^2}}\right)\quad\text{.....................(5).}$$

From this equation, if we know Y, the resistance that a given curve may be described can be found.

225. If the resistance vary as the product of the density into the n^{th} power of the velocity, equations (4) and (5) may be used to find the law of density that a given curve may be described. For in this case R will be represented by $k\left(\frac{ds}{dt}\right)^n$, where k is proportional to the density at any point, and is to be determined as a function of x and y. We have then from (5)

$$k\left(\frac{ds}{dx}\right)^n\left(\frac{dx}{dt}\right)^n=-\frac{1}{2}\frac{ds}{dx}\frac{d}{dx}\left(\frac{Y}{\frac{d^2y}{dx^2}}\right);$$

and hence, by (4),

$$k = -\frac{1}{2}\left(\frac{ds}{dx}\right)^{1-n}\left(\frac{Y}{\frac{d^2y}{dx^2}}\right)^{-\frac{n}{2}}\frac{d}{dx}\left(\frac{Y}{\frac{d^2y}{dx^2}}\right),$$

which, according as n is not or is equal to 2, may be written

$$k = \frac{1}{n-2}\left(\frac{ds}{dx}\right)^{1-n}\frac{d}{dx}\left(\frac{Y}{\frac{d^2y}{dx^2}}\right)^{\frac{2-n}{2}} \quad\ldots\ldots\ldots\ldots (6);$$

$$\text{or } k = -\frac{1}{2}\frac{d}{ds}\log\frac{Y}{\frac{d^2y}{dx^2}} \quad\ldots\ldots\ldots\ldots (7).$$

226. Equations (5) and (6) may also clearly be used to determine the laws of force by which a given curve may be described, the density being supposed known, and the direction of the force being given.

From equation (3), we obtain

$$\frac{\left(\frac{ds}{dt}\right)^3}{Y\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{ds}{dt}\right)^3}{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}} = \rho,$$

ρ being the radius of curvature. Hence

$$\left(\frac{ds}{dt}\right)^2 = Y\rho\frac{dx}{ds}$$

$$= 2Y\,(\tfrac{1}{4}\text{ chord of curvature parallel to } y),$$

and therefore the velocity at any point is the same as would be acquired by a particle falling in vacuo through one fourth of the chord of curvature parallel to the fixed line under the action of a uniform force equal to the force at that point.

227. *A particle, moving in a resisting medium, is acted on by a central force; to find the resistance that a given curve may be described.*

P denoting the central force and R the resistance, the equations of motion are

$$\frac{d^2x}{dt^2} = -P\frac{x}{r} - R\frac{dx}{ds},$$

$$\frac{d^2y}{dt^2} = -P\frac{y}{r} - R\frac{dy}{ds}.$$

Eliminating R,

$$\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2} = \frac{P}{r}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right),$$

$$\text{or } \frac{1}{\rho}\left(\frac{ds}{dt}\right)^3 = P\frac{p}{r}\frac{ds}{dt},$$

$$\text{hence } \left(\frac{ds}{dt}\right)^2 = Pp\frac{dr}{dp} \quad\text{.........................}\quad (1),$$

which shews that the velocity is that due to one fourth the chord of curvature through the center.

Again, eliminating P from the equations of motion, we obtain

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = -R\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\frac{dt}{ds},$$

$$\text{or } \frac{d}{dt}\left(p\frac{ds}{dt}\right) = -Rp;$$

$$\text{whence } p\frac{ds}{dt}\frac{d}{ds}\left(p\frac{ds}{dt}\right) = -Rp^2;$$

$$\text{therefore } -2Rp^2 = \frac{d}{ds}\left\{p^2\left(\frac{ds}{dt}\right)^2\right\}$$

$$= \frac{d}{ds}\left(Pp^3\frac{dr}{dp}\right), \text{ by (1)};$$

$$\text{and finally } R = -\frac{1}{2p^2}\frac{d}{ds}\left(Pp^3\frac{dr}{dp}\right) \quad\text{..................}\quad (2),$$

which determines the required law of resistance.

228. If the resistance vary as the product of the density and the n^{th} power of the velocity,

$$R = k\left(\frac{ds}{dt}\right),$$

and by equations (1) and (2),

$$k\left(Pp\frac{dr}{dp}\right)^{\frac{n}{2}} = -\frac{1}{2p^2}\frac{d}{ds}\left(Pp^3\frac{dr}{dp}\right),$$

$$\therefore\ k = -\frac{p^{n-2}}{2}\left(Pp^3\frac{dr}{dp}\right)^{-\frac{n}{2}}\frac{d}{ds}\left(Pp^3\frac{dr}{dp}\right),$$

which, according as n is not or is equal to 2, may be written

$$k = \frac{p^{n-2}}{n-2}\frac{d}{ds}\left(Pp^3\frac{dr}{dp}\right)^{\frac{2-n}{2}},$$

$$\text{or } k = -\frac{1}{2}\frac{d}{ds}\log\left(Pp^3\frac{dr}{dp}\right).$$

These equations determine either the law of density that a given curve may be described, the force being supposed known, or the law of force supposing the density known.

229. *A particle, moving in a resisting medium of which the resistance varies as the product of the density and the square of the velocity, is acted on by a central force; to determine the orbit.*

This may be derived immediately from the result of last Article, but we will here give a direct investigation. Let P be the central force; then, taking the equations of motion along the radius vector, and perpendicular to it, we have

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P - k\left(\frac{ds}{dt}\right)^2\frac{dr}{ds} \ldots\ldots\ldots\ldots (1),$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = -k\left(\frac{ds}{dt}\right)^2\frac{rd\theta}{ds}.$$

Dividing the latter of these equations by $r\dfrac{d\theta}{dt}$, we have

$$\frac{\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)}{r^2\frac{d\theta}{dt}} = -k\frac{ds}{dt}.$$

Integrating,

$$r^2\frac{d\theta}{dt} = h\epsilon^{-\int k ds};$$

h being the constant introduced in the integration and depending on the initial circumstances.

Again, putting $r = \frac{1}{u}$,

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$= -h\frac{du}{d\theta}\epsilon^{-\int k ds};$$

$$\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\epsilon^{-\int k ds}\frac{d\theta}{dt} + hk\frac{du}{d\theta}\epsilon^{-\int k ds}\frac{ds}{dt}$$

$$= -h^2u^2\frac{d^2u}{d\theta^2}\epsilon^{-2\int k ds} - k\frac{ds}{dt}\frac{dr}{dt}.$$

Substituting in equation (1), we have, since

$$\left(\frac{ds}{dt}\right)^2\frac{dr}{ds} = \frac{ds}{dt}\frac{dr}{dt},$$

$$-h^2u^2\frac{d^2u}{d\theta^2}\epsilon^{-2\int k ds} - h^2u^3\epsilon^{-2\int k ds} = -P;$$

and finally $\frac{d^2u}{d\theta^2} + u - \frac{P\epsilon^{2\int k ds}}{h^2u^2} = 0.$

EXAMPLES.

(1) A particle is projected with a given velocity V in a uniform medium in which the resistance varies as the square

root of the velocity; to find what time will elapse before the particle is reduced to rest.

$$\text{Required time} = \frac{2V^{\frac{1}{2}}}{k}.$$

(2) A particle projected with a velocity of 1000 feet a second, loses half its velocity by passing through 3 inches of a resisting medium of which the resistance is uniform; to find the time of passing through this space.

$$\frac{1}{3000}\text{th of a second.}$$

(3) A particle falls towards a center of force which varies as the inverse cube of the distance, in a medium of which the density varies also as the inverse cube, and of which the resistance varies as the square of the velocity; prove that at any distance x from the center,

$$(\text{velocity})^2 = \frac{\mu}{k}\left\{1 - \epsilon^{-k\left(\frac{1}{x^2} - \frac{1}{a^2}\right)}\right\},$$

where μ = force at unit of distance, k = density at unit of distance, and a = distance of the particle from the center at the beginning of its motion.

The equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^3} + \frac{k}{x^3}\left(\frac{dx}{dt}\right)^2;$$

$$\text{or } v\frac{dv}{dx} = -\frac{\mu}{x^3} + \frac{kv^2}{x^3}.$$

Let $v^2 = z$;

$$\therefore \frac{dz}{dx} - \frac{2kz}{x^3} = -\frac{2\mu}{x^3}.$$

Multiply by $\epsilon^{\frac{k}{x^2}}$ and integrate;

$$\therefore \epsilon^{\frac{k}{x^2}} z = C + \frac{\mu}{k}\epsilon^{\frac{k}{x^2}}.$$

Now when $x=a$, $z=v^2=0$; $\therefore C=-\frac{\mu}{k}\epsilon^{\frac{k}{a^2}}$;

$$\therefore z=v^2=\frac{\mu}{k}\left\{1-\epsilon^{-k\left(\frac{1}{x^2}-\frac{1}{a^2}\right)}\right\}.$$

(4) A particle acted on by a constant force f, moves from rest in a medium of which the resistance varies as the square of the velocity directly and as the distance from the center inversely; find the velocity of the particle at any distance from the origin, and the position of the particle when its velocity is a maximum.

$$x^{-2k}v^2=\frac{2f}{1-2k}(a^{1-2k}-x^{1-2k}),\quad x'=(2k)^{\frac{1}{1-2k}}a,$$

where a is the initial distance and k the resistance when x and v are both unity.

(5) If chords be drawn from either extremity of a vertical diameter of a circle, the time of descent down each of them in a medium whose resistance varies as the velocity, is the same.

(6) A particle is projected with a given velocity, towards a center of force attracting inversely as the cube of the distance, in a medium of which the resistance varies as the square of the velocity directly, and as the square of the distance from the center inversely; to find the velocity at any point.

$$\text{Here}\quad \epsilon^{-\frac{2k}{a}}v^2-\epsilon^{-\frac{2k}{x}}V^2=\frac{\mu}{2k^2}\left(\frac{a-2k}{a}\epsilon^{-\frac{2k}{x}}-\frac{x-2k}{x}\epsilon^{-\frac{2k}{a}}\right),$$

where V is the velocity of projection, μ the absolute attracting force, k the resistance at a unit of distance, a the initial distance, and x the distance of the particle corresponding to the velocity v.

(7) One particle begins to fall from the higher extremity of a vertical line, and at the same instant another is projected upwards from the other extremity with a given velocity, the particles moving in a medium of which the resistance varies directly as the velocity; shew that the time at which they

will meet $=\frac{1}{k}\log\frac{V}{V-ka}$, where a is the length of the vertical line, V the velocity of projection, and k the resistance for a unit of velocity.

(8) A particle acted on by gravity falls from a given altitude in an uniform medium of which the resistance varies as the square of the velocity; on arriving at the lowest point of its descent it is reflected upwards with the velocity which it has acquired in its fall, after reaching its greatest altitude it again descends and is again reflected, and so on perpetually; to determine the altitude of ascent after any number of reflections.

Let a be the height from which the particle falls, a_1, a_2, a_3, &c., its maximum altitudes afterwards, v_1, v_2, v_3, &c., its velocities after the first, second, &c. times of reaching the bottom, then (§ 217),

$$v_1^2 = K^2(1-\epsilon^{-\frac{2fa}{K^2}}),$$

and

$$\frac{1}{v_2^2}-\frac{1}{v_1^2}=\frac{1}{K^2},$$

$$\frac{1}{v_3^2}-\frac{1}{v_2^2}=\frac{1}{K^2},$$

....

....

$$\frac{1}{v_n^2}-\frac{1}{v^2_{n-1}}=\frac{1}{K^2}.$$

Adding

$$\frac{1}{v_n^2}-\frac{1}{v_1^2}=\frac{n-1}{K^2}.$$

Hence

$$\frac{1}{v_n^2}=\frac{1}{K^2}\left(n-1+\frac{1}{1-\epsilon^{-\frac{2fa}{K^2}}}\right)$$

$$=\frac{1}{K^2}\left\{\frac{n-(n-1)\epsilon^{-\frac{2fa}{K^2}}}{1-\epsilon^{-\frac{2fa}{K^2}}}\right\};$$

$$\therefore \frac{v_n^2}{K^2} = \frac{\epsilon^{\frac{2fa}{K^2}} - 1}{n\epsilon^{\frac{2fa}{K^2}} - n + 1}.$$

The particle is now projected upwards with a velocity v_n, therefore (§ 217)

$$a_n = \frac{K^2}{2f} \log\left(1 + \frac{v_n^2}{K^2}\right)$$

$$= \frac{K^2}{2f} \log \frac{(n+1)\,\epsilon^{\frac{2fa}{K^2}} - n}{n\epsilon^{\frac{2fa}{K^2}} - n + 1}.$$

If a be equal to infinity,

$$a_n = \frac{K^2}{2f} \log \frac{n+1}{n}.$$

(9) To determine the law of force that a particle may always descend to a given center in the same time from whatever distance it commences its motion, the medium in which the particle moves being uniform, and the resistance varying as the square of the velocity.

(10) If one particle be projected in a medium, the resistance of which varies as the velocity, and be acted on by a constant force parallel to a given line, and another be projected in vacuo at the same angle and with the same velocity and be acted on by the same constant force, and if t_1, t_2 be the times of describing two arcs in the medium and in vacuo so related to each other that the tangents at their extremities shall be parallel to each other, then $\epsilon^{kt_1} - 1 = kt_2$, k being the resistance corresponding to the velocity 1.

(11) A particle moves in a semicircle, acted on by a constant force in parallel lines; find the requisite resistance, and supposing the resistance to vary as the density into the square of the velocity, find the law of density.

Let the equation to the circle be $x^2 + y^2 = a^2$, and the force parallel to y; then $R \propto x$, $k \propto \frac{x}{(a^2 - x^2)^{\frac{1}{2}}}$.

(12) A particle acted on by a constant force parallel to the axis of y moves in the curve $a^{n-1}y + x^n = b^n$; find the law of resistance.

$$R \propto \frac{\sqrt{(a^{2n-2} + n^2 x^{2n-2})}}{x^{n-1}}.$$

(13) A particle acted on by a constant force parallel to the axis of y moves in the curve, $y = mx + \frac{a^{n+1}}{x^n}$; find the law of density, the resistance varying as the product of the density and square of the velocity.

$$k \propto \frac{x^n}{\{x^{2n+2} + (mx^{n+1} - na^{n+1})^2\}^{\frac{1}{2}}}.$$

(14) A particle moves in a circle about a center of force in the circumference, the force being attractive and $= \mu r^n$; to find the resistance of the medium and the law of the density, supposing the resistance equal to the product of the density and the square of the velocity.

$$R = \frac{\mu}{4}(n+5)\, r^n \sin\theta, \quad k = \frac{n+5}{2}\frac{\sin\theta}{r}.$$

(15) A particle moves towards the pole in an equiangular spiral about a center of force in the pole, the force being μr^n; to find the resistance and density of the medium, the resistance being equal to the product of the density and the square of the velocity.

$$R = \frac{\mu(n+3)}{2} r^n \cos\alpha, \quad k = \frac{n+3}{2}\frac{\cos\alpha}{r},$$

where α is the constant angle of the spiral.

(16) A particle moves in the circumference of a circle about a center of force in the center, the resistance of the medium in which the motion takes place being constant; to find the law of force, the velocity at any time, and the time which elapses as well as the space described before the particle is reduced to rest.

$$v^2 = V^2 - 2cs, \quad P = \frac{1}{a}(V^2 - 2cs), \quad S = \frac{V^2}{2c}, \quad T = \frac{V}{c},$$

where V = initial velocity, c = constant force of resistance, a = radius of the circle, s the arc described from the beginning of the motion, S and T the space and time corresponding to the particle's being reduced to rest.

(17) A particle is projected along a smooth circle with velocity V in a medium whose resistance $\propto v^2$. Prove that when the direction of the motion has changed through an angle ϕ the velocity $= V\epsilon^{-k\phi}$.

(18) A particle moves towards the pole in an equiangular spiral, the motion taking place in a medium whose resistance $= kr^n$; find the law of central attractive force in the pole.

$$P = \frac{(n+3)\cos\alpha\, a^2 V^2 + 2k\,(r^{n+3} - a^{n+3})}{(n+3)\cos\alpha\, r^3},$$

where V = initial velocity, and α = constant angle of the spiral.

(19) If a particle acted on by a central force P is moving in a medium whose resistance $= k$ (velocity) prove that

$$\frac{d^2r}{dt^2} + P - \frac{h^2}{r^3}\epsilon^{-2kt} + k\frac{dr}{dt} = 0,$$

where h is an arbitrary constant.

(20) A heavy particle moving in a medium whose resistance $= nv^2$, is compelled to describe in a vertical plane the curve

$$ax = \epsilon^{ns} - ns - 1,$$

where s is the length of the curve measured from the lowest point, x the abscissa of the extremity of this arc referred to a vertical axis, and a a constant; shew that the time of reaching the lowest point is independent of the height from which it starts.

(21) Shew that the curve in last question is also tautochronous if the resistance $= mv + nv^2$.

CHAPTER IX.

GENERAL THEOREMS.

230. We propose now to prove some of the general theorems connected with the motion of a particle under the action of any forces, and to investigate the forces requisite for the description of given paths in a given manner. Several of these results have already occurred as immediate deductions from the laws of motion; but to maintain the special character of the work we give more formal analytical demonstrations, though these are certainly superfluous.

231. *If a particle be subject to the action of forces, whose resultant is continually at right angles to the direction of its motion; the velocity will be uniform.*

Let R be this resultant, λ, μ, ν, its direction cosines, then if the mass of the particle be taken as unit,

$$\frac{d^2x}{dt^2} = \lambda R,$$

$$\frac{d^2y}{dt^2} = \mu R,$$

$$\frac{d^2z}{dt^2} = \nu R.$$

Multiplying by $\frac{dx}{dt}$, ..., adding, and observing that

$$\lambda \frac{dx}{ds} + \mu \frac{dy}{ds} + \nu \frac{dz}{ds} = 0$$

since the force R is at right angles to the element of the path,

$$\frac{1}{2}\frac{d}{dt}(v^2) = \frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2} = 0;$$

or $v = \text{const.}$

Or, we might at once have resolved along the arc; this would have given

$$\frac{d^2s}{dt^2} = 0;$$

whose integral is

$$\frac{ds}{dt} = v = \text{const.}$$

The value of R (§§ 16, 64) is evidently $\frac{v^2}{\rho}$; or varies inversely as the radius of absolute curvature of the path; and it is clear that its direction lies in the osculating plane, since there is no acceleration perpendicular to that plane.

232. Ex. I. *A particle projected in a plane is acted on by a constant force* R *in that plane continually perpendicular to its direction of motion; to find the path described.*

Here $R = \frac{v^2}{\rho}$; and therefore ρ is constant, or the path is a circle.

Ex. II. *Let* R *vary as the time elapsed since the commencement of the motion; then* $R = R_0 t$.

Also s, the arc of the curve described in the same time, $= vt$, since v is constant.

Hence we have

$$R = \frac{v^2}{\rho} = R_0 \frac{s}{v};$$

$$\text{or } \rho s = \text{const.} = c^2, \text{ suppose.}$$

If ψ be the angle which the direction of the element δs makes with any fixed line,

$$\rho = \frac{ds}{d\psi};$$

$$\text{and therefore } s\frac{ds}{d\psi} = c^2,$$

$$\text{or } s^2 = 2c^2 (\psi + C),$$

the intrinsic equation to the curve described.

233. *If* X, Y, Z *be the rectangular components of a force or forces such as occur in nature, i.e. tending to fixed centers and being functions of the distances from these centers,*

$$\mathrm{X}d\mathrm{x} + \mathrm{Y}d\mathrm{y} + \mathrm{Z}d\mathrm{z} = -d\mathrm{V},$$

i.e. is a complete differential. Compare § 73.

Let the points a_1, b_1, c_1; a_2, b_2, c_2; &c., be the positions of the centers of force; x, y, z the co-ordinates of the attracted particle; then, if $r_1, r_2, \ldots$ be its distances from the centers,

$$\phi_1'(D),\ \phi_2'(D),\ \&c.,$$

the laws of attraction to those centers; we have

$$X = \frac{a_1 - x}{r_1}\phi_1'(r_1) + \frac{a_2 - x}{r_2}\phi_2'(r_2) + \ldots\ldots\ldots$$

$$= \Sigma \frac{a-x}{r}\phi'(r).$$

But $r = \sqrt{\{(a-x)^2 + (b-y)^2 + (c-z)^2\}}$;

which gives $\left(\frac{dr}{dx}\right) = -\frac{a-x}{r}$, &c., for the values of the partial differential coefficients of r.

Hence,

$$X = -\Sigma\phi'(r)\left(\frac{dr}{dx}\right),$$

$$Y = -\Sigma\phi'(r)\left(\frac{dr}{dy}\right),$$

$$Z = -\Sigma\phi'(r)\left(\frac{dr}{dz}\right).$$

These give

$$Xdx + Ydy + Zdz$$

$$= -\Sigma\phi'(r)\left\{\left(\frac{dr}{dx}\right)dx + \left(\frac{dr}{dy}\right)dy + \left(\frac{dr}{dz}\right)dz\right\}$$

$$= -\Sigma\phi'(r)\,dr = -dV \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

since every term of the sum is a complete differential. From § 73 it is obvious that V is the potential energy of unit of matter at x, y, z.

234. *Under the action of any forces such as occur in nature the increment of the square of the velocity of a particle in passing from one point to another is independent of the path pursued, and depends only on the initial and final positions. This is true even if the particle be forced to move in any particular path by a force continually perpendicular to its direction of motion, such as frictionless constraint.*

Taking tangential resolution, the force of constraint disappears, and

$$\frac{d^2s}{dt^2} = S.$$

Now S, the resolved force along the tangent, is

$$X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds},$$

and therefore by (1),

$$\frac{ds}{dt}\frac{d^2s}{dt^2} = -\Sigma\phi'(r)\frac{dr}{dt} = -\frac{dV}{dt},$$

$$\text{or } \frac{v^2}{2} = C - \Sigma\phi(r) = C - V;$$

hence, if U be the velocity at a point whose distances from the centers are $R_1, R_2, \ldots\ldots$, and where $V = V_1$,

$$\frac{v^2}{2} - \frac{U^2}{2} = \Sigma\phi(R) - \Sigma\phi(r) = V_1 - V,$$

a result which involves only the co-ordinates of the initial and final positions. See, again, § 73.

235. Hence if from any point of the surface

$$V = \Sigma\phi(r) = A,$$

a particle be projected with a given velocity in any direction; its velocity when it meets the surface

$$V = \Sigma\phi(r) = B,$$

will be the same, in whatever point it meet that surface; A and B being any constants.

Now on account of equation (1), $V = \Sigma\phi(r) =$ constant is the equation to a surface on which if smooth a particle will rest in any position under the action of the given forces.

Hence a particle leaving any point of a surface of equilibrium with a given velocity, will have on reaching any other surface of equilibrium a velocity independent of the path pursued or the point reached. This is evident from § 73 if we notice that a surface of equilibrium is an *Equipotential Surface.*

236. *To find the condition to which the applied forces must be subject when the vis viva of a particle depends upon its position only.* This is merely the converse of § 234.

Here we have

$$\frac{1}{2}v^2 = \phi(x, y, z),$$

and, therefore,

$$vdv = \left(\frac{d\phi}{dx}\right)dx + \left(\frac{d\phi}{dy}\right)dy + \left(\frac{d\phi}{dz}\right)dz.$$

But, in all cases of motion,

$$vdv = Xdx + Ydy + Zdz.$$

Hence, in this case we must have

$$X = \left(\frac{d\phi}{dx}\right), \quad Y = \left(\frac{d\phi}{dy}\right), \quad Z = \left(\frac{d\phi}{dz}\right);$$

that is,

$$Xdx + Ydy + Zdz$$

must be a complete differential of three independent variables.

If the seat of the force be in a definite fixed point, which may be taken as origin, the velocity must evidently depend solely on the *distance* from that point; hence, if

$$r = \sqrt{x^2 + y^2 + z^2},$$

we have

$$\frac{1}{2}v^2 = \phi(r).$$

The above process gives, in this case,

$$vdv = Xdx + Ydy + Zdz = d\phi(r)$$

$$= \phi'(r)\left(\frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz\right),$$

$$\text{or } \frac{X}{x} = \frac{Y}{y} = \frac{Z}{z},$$

which shew that *the force is in the direction of r.* From this it evidently follows that *its magnitude must be a function of r.*

237. The proposition of § 234 contains the Conservation of Energy; or, as it was formerly called, the *Principle of Vis Viva;* for the case of a single particle.

From this principle it follows that if several particles moving under the action of the same center of force have equal velocities at any particular distance from their center; their velocities will always be equal at equal distances from that center.

Now we have seen (§ 142) that the axis major, $2a$, of an elliptic orbit about a center of force in the focus is independent of the direction of projection. Hence, by considering the projection to be made *from* the center, we find that the velocity at any point is due to a fall, from rest at a distance $2a$, to that point; and that, therefore, in any elliptic orbit about a focus the velocity at any point is that due to a fall to the point, through a space equal to the distance from the other focus.

238. *If the forces acting on a particle, and the square of its velocity, be increased at any instant in the same ratio, the path will not be altered.*

For the tangent, and the osculating plane, which contains the tangent and the resultant force, are evidently not altered. And the curvature, being

$$\frac{\text{Normal Component of Forces}}{\text{Square of velocity}}, \text{ § } 64,$$

has its numerator and denominator increased in the same ratio. And the square of the velocity at the end of any arc is in-

creased in the same ratio as that at the beginning. Hence each successive elementary arc of the path remains unchanged.

239. *If a number of separate particles whose masses are* m_1, m_2, *&c. subjected to the action of forces* f_1, f_2, *&c. respectively, and projected from the same point in the same direction with velocities* $\overline{v}_1$, $\overline{v}_2$, *&c., all describe one path; the same path will also be described by a particle of mass* M *projected with velocity* $\overline{U}$ *from the same point in the same direction, and acted on at once by the same forces* f_1, f_2, *&c. provided* $M\overline{U}^2 = \Sigma (m\overline{v}^2)$.

Suppose that, in addition to the forces f_1, f_2, &c., a force R continually acting in a direction at right angles to that of M's motion be required to cause it to move in the given path; i.e. suppose M to be constrained by a smooth tube to move in the required path; the equations of motion are

$$M\frac{d^2x}{dt^2} = \Sigma (X) + R\lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

with similar equations in y and z,

where λ, μ, ν are the direction cosines of R, and X, Y, Z the resolved parts of f.

Multiplying by $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ in order, and adding, R goes out, and we have

$$\frac{1}{2} Md(U^2) = \Sigma (X)\, dx + \Sigma (Y)\, dy + \Sigma (Z)\, dz.$$

But for the separate particles m_1, m_2, &c. we have

$$\frac{1}{2} m_1 d(v_1^2) = X_1 dx + Y_1 dy + Z_1 dz, \text{ \&c.}$$

therefore, the path being the same for all,

$$\frac{1}{2}\Sigma \{md(v^2)\} = \Sigma (X)\, dx + \Sigma (Y)\, dy + \Sigma (Z)\, dz.$$

Hence $\Sigma \{md(v^2)\} = Md(U^2)$,

or $\Sigma (mv^2) = MU^2 + C$.

But $\Sigma (m\overline{v}^2) = M\overline{U}^2$ by hypothesis, therefore $C = 0$.

[Instead of this analysis, it is sufficient (by § 73) to notice that the work done on M is the sum of that done on m_1, m_2, &c. Hence the *increase* of kinetic energy must be the same; and if, at starting, the kinetic energy of M be the sum of those of m_1, m_2, &c. it will remain so throughout the motion.]

Hence the vis viva of M will be at each point of the orbit equal to the sum of the vires vivæ of m_1, m_2, &c., at that point. To find R, notice that in general the pressure on a constraining curve is the sum of, the resolved parts of the impressed forces, and the pressure due to the velocity. Now the latter part is as the vis viva, therefore, in the case of M it is the sum of the corresponding forces in the case of m_1, m_2, &c. Also the same may be said of the resolved parts of the impressed forces. But in the case of each particle, these partial pressures destroyed each other, since the curve was described freely, hence their sums will destroy each other, or the curve will be freely described by M.

240. *If at any instant the velocity of a material particle, moving under the action of a conservative system of forces*, § 72, *be reversed, the particle will describe its former path in the reverse direction.* [Compare Ex. (48) to Chap. V.]

Suppose a smooth tube, in the form of the original path, be requisite to constrain the particle to move backwards along it. The velocity will be, at each point, of the same magnitude as before, (§ 234); the resultant force, and the curvature of the path, also alike; hence the normal component of the force will produce the requisite curvature of the path, and there will be no pressure on the constraining tube. The tube is, therefore, not required. Whence the proposition.

241. Least, or Stationary, Action. If v be the velocity of a particle whose mass is m, and if s be the arc of the path described, the value of the integral $m\int v ds$ taken between proper limits is called the *Action* of the particle.

If a particle move freely, or on a smooth surface, (under the action of forces such as occur in nature,) between any two points, the value of the integral $m\int vds$ *for the whole actual path is generally less than it would be if the particle were constrained to pass from one point to the other by a different path.*

This, combined with the above definition, is for a single particle the *Principle of Least Action;* of which in an elementary work like the present we can give only a very imperfect sketch. For farther information see Thomson and Tait's *Natural Philosophy*, § 318.

242. If δ be the symbol of the Calculus of Variations, the proposition will be proved if we shew that

$$\delta A = \delta \int v ds = 0.$$

Now $\delta \int v ds = \int \delta (v ds) = \int (v \delta ds + ds \delta v)$

$$= \int (v \delta ds + dt v \delta v), \text{ since } v = \frac{ds}{dt}.$$

But generally, $\frac{1}{2} v^2 = \int (X dx + Y dy + Z dz) = \psi (x, y, z)$,

the force of constraint, if any, having disappeared;

$$\text{hence } v \delta v = X \delta x + Y \delta y + Z \delta z.$$

But (§ 191) $$X = \frac{d^2 x}{dt^2} - R\lambda, \text{ \&c.}$$

Hence

$$v \delta v = \left(\frac{d^2 x}{dt^2} \delta x + \frac{d^2 y}{dt^2} \delta y + \frac{d^2 z}{dt^2} \delta z \right) - R (\lambda \delta x + \mu \delta y + \nu \delta z).$$

Now if the particle remain on the surface whose equation is $F = 0$,

$$\lambda \delta x + \mu \delta y + \nu \delta z = k \delta F = 0,$$

and if it leave it $R = 0$, so in either case the latter term on the right vanishes.

$$\text{Also } ds^2 = dx^2 + dy^2 + dz^2;$$

which gives $ds \delta ds = dx \delta dx + dy \delta dy + dz \delta dz$,

$$\text{or } v\delta ds = \frac{dx}{dt}\delta dx + \frac{dy}{dt}\delta dy + \frac{dz}{dt}\delta dz$$

$$= \frac{dx}{dt} d\delta x + \frac{dy}{dt} d\delta y + \frac{dz}{dt} d\delta z,$$

since the order of d and δ is immaterial.

Hence

$$\delta A = \delta\int v ds = \int\left\{\frac{dx}{dt} d\delta x + \frac{dy}{dt} d\delta y + \frac{dz}{dt} d\delta z + \delta x d\left(\frac{dx}{dt}\right) + \delta y d\left(\frac{dy}{dt}\right) + \delta z d\left(\frac{dz}{dt}\right)\right\}$$

$$= \left[\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right],$$

taken between proper limits. Now at both limits

$$\delta x = 0, \quad \delta y = 0, \quad \delta z = 0;$$

hence we have $\delta A = 0$.

243. It is commonly said that as, in general, it is impossible to suppose the action a maximum, this result shews that it is a minimum. The true interpretation of the expression, $\delta A = 0$, is that the unconstrained path of the particle is such, that a small deviation from it will produce an infinitely smaller change in the value of A. Hence Hamilton has suggested the more appropriate title *Stationary Action*.

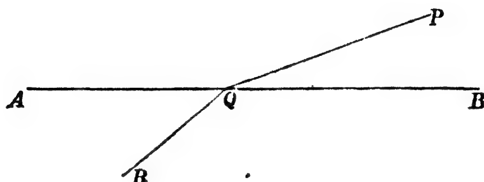
244. If no forces act on the particle except the constraint of the surface, we have v constant, and the above equation shews that in this case the length of the path is generally a minimum.

A particle therefore, projected along a smooth surface and subject to no forces, will trace out between any two points in its path the shortest line on the surface. (§ 193).

It may happen, in the case of a sphere for instance, that the particle will not trace out the shortest line on the surface between the two points; but we cannot here enter into the details which are necessary to the full elucidation of such cases.

245. We may apply this principle directly to form the equations of motion in any particular case, or to find the actual path under the action of any forces.

Ex. I. *Let us take again the case of the refraction of light in the corpuscular theory.* (§ 122).



The velocity in the upper medium is supposed to be u, that in the lower v.

In this case the expression for the action becomes simply

$$uPQ + vQR,$$

if PQR be the path of the particle.

By making this quantity a minimum, as depending on the position of Q, P and R being given points; it is easy to shew that Q must lie in the plane through P and R perpendicular to the surface AB, and also that the sines of the inclinations of PQ and QR to the normal at Q must be in the inverse ratio of the velocities.

246. Ex. II. *To find the equation of the path described by a particle about a center of force.*

Let P be the central force at distance r, then

$$v^2 = C - 2\int P dr, \text{ (§ 134)}$$

$$= \{\phi(r)\}^2, \text{ suppose, ...(1)},$$

which gives

$$\int v ds = \int \phi(r) ds.$$

Hence

$$
\begin{aligned}
0 &= \delta\int\phi(r)\,ds \\
&= \int\{\phi'(r)\,\delta r\,ds + \phi(r)\,d\delta s\} \\
&= \int\left\{\frac{\phi'(r)}{r}(x\delta x + y\delta y + z\delta z)\,ds \right.\\
&\qquad\left. + \phi(r)\left(\frac{dx}{ds}\,d\delta x + \frac{dy}{ds}\,d\delta y + \frac{dz}{ds}\,d\delta z\right)\right\} \\
&= \left[\phi(r)\left(\frac{dx}{ds}\,\delta x + \frac{dy}{ds}\,\delta y + \frac{dz}{ds}\,\delta z\right)\right] \\
&\quad + \int\left[\frac{\phi'(r)}{r}(x\delta x + y\delta y + z\delta z)\,ds \right.\\
&\qquad\left. - \delta x d\left\{\phi(r)\frac{dx}{ds}\right\} - \delta y d\left\{\phi(r)\frac{dy}{ds}\right\} - \delta z d\left\{\phi(r)\frac{dz}{ds}\right\}\right].
\end{aligned}
$$

The integrated part refers only to the limits, and must therefore vanish independently of the integral. That the integral may be identically zero, we must have

$$\frac{x\phi'(r)}{r} - \frac{d}{ds}\left\{\phi(r)\frac{dx}{ds}\right\} = 0,$$

with similar equations in y and z. These may be written

$$
\left.
\begin{aligned}
\phi'(r)\left(\frac{x}{r} - \frac{dr}{ds}\frac{dx}{ds}\right) - \phi(r)\frac{d^2x}{ds^2} = 0 \\
\phi'(r)\left(\frac{y}{r} - \frac{dr}{ds}\frac{dy}{ds}\right) - \phi(r)\frac{d^2y}{ds^2} = 0 \\
\phi'(r)\left(\frac{z}{r} - \frac{dr}{ds}\frac{dz}{ds}\right) - \phi(r)\frac{d^2z}{ds^2} = 0
\end{aligned}
\right\} \dots\dots\dots\dots (a).
$$

Multiplying by any three constants, A, B, C, and adding, we have

$$
\begin{aligned}
&(Ax + By + Cz)\frac{\phi'(r)}{r} \\
&- \left(A\frac{dx}{ds} + B\frac{dy}{ds} + C\frac{dz}{ds}\right)\phi'(r)\frac{dr}{ds} \\
&- \left(A\frac{d^2x}{ds^2} + B\frac{d^2y}{ds^2} + C\frac{d^2z}{ds^2}\right)\phi(r) = 0;
\end{aligned}
$$

which is obviously satisfied by

$$Ax + By + Cz = 0,$$

which shews that the orbit is in a plane passing through the center of force. Let x, y be this plane, then we may confine ourselves to the first two of equations (a).

Multiplying the second by x and the first by y and subtracting, we obtain

$$\phi'(r)\frac{dr}{ds}\left(x\frac{dy}{ds} - y\frac{dx}{ds}\right) + \phi(r)\left(x\frac{d^2y}{ds^2} - y\frac{d^2x}{ds^2}\right) = 0.$$

This is immediately integrable, and gives

$$\phi(r)\left(x\frac{dy}{ds} - y\frac{dx}{ds}\right) = \text{constant}.$$

Since $\phi(r) = v$, we see by § 22 that this is in polar co-ordinates

$$r^2\frac{d\theta}{dt} = h \text{ .... } (b),$$

which is the equation for the equable description of areas.

Finally, multiplying these two first equations of group (a) by x and y respectively and adding, we have

$$r\phi'(r)\left\{1 - \left(\frac{dr}{ds}\right)^2\right\} - \phi(r)\left(x\frac{d^2x}{ds^2} + y\frac{d^2y}{ds^2}\right) = 0 \text{ .... } (c).$$

But, since

$$r\frac{dr}{ds} = x\frac{dx}{ds} + y\frac{dy}{ds},$$

we have by differentiation

$$x\frac{d^2x}{ds^2} + y\frac{d^2y}{ds^2} = r\frac{d^2r}{ds^2} + \left(\frac{dr}{ds}\right)^2 - 1.$$

Substituting in (c), and changing the independent variable from s to θ by means of the equation

$$ds^2 = dr^2 + r^2d\theta^2,$$

we have

$$\phi'(r)\,r\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}-\phi(r)\left\{r\frac{d^2r}{d\theta^2}-2\left(\frac{dr}{d\theta}\right)^2-r^2\right\}=0.$$

Putting $\frac{1}{u}$ for r, this becomes

$$\frac{d^2u}{d\theta^2}+u=-\frac{\phi'(r)}{\phi(r)}\left\{1+\frac{1}{u^2}\left(\frac{du}{d\theta}\right)^2\right\}\ldots\ldots\ldots\ldots\ldots(d).$$

But, by (*b*) as developed in § 133,

$$v^2=\{\phi(r)\}^2=h^2\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\}.$$

Also $\phi(r)\,\phi'(r)=-P$, by (1).

Thus (*d*) becomes

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2},\text{ as in § 127.}$$

247. We might have treated these equations (§ 246 (*a*)) somewhat differently thus

$$\phi(r)=v=\frac{ds}{dt}.$$

$$\text{Hence }\phi(r)\frac{dx}{ds}=\frac{dx}{dt},\text{ \&c.;}$$

and we have the equations

$$\frac{x\phi'(r)}{r}-\frac{d}{ds}\left(\frac{dx}{dt}\right)=0,\text{ \&c. \&c.,}$$

which give, at once,

$$\frac{d\left(\frac{dx}{dt}\right)}{x}=\frac{d\left(\frac{dy}{dt}\right)}{y}=\frac{d\left(\frac{dz}{dt}\right)}{z},$$

containing the theorems of constant plane and equable description of areas; and since

$$\phi'(r)\frac{ds}{dt}=\phi(r)\,\phi'(r)=-P,$$

$$-\frac{x}{r}P - \frac{d^2x}{dt^2} = 0, \text{ \&c.,}$$

the ordinary equations in three rectangular directions.

248. We might have simplified the work by using polar co-ordinates immediately after having proved that the orbit is plane. For we have

$$A = \int \phi(r) \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} d\theta, \text{ a minimum,}$$

and therefore (by the formula $V = Pp + C$)

$$\phi(r) \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} = \phi(r) \frac{\left(\frac{dr}{d\theta}\right)^2}{\sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}} + C,$$

or reducing, and putting h for C,

$$r^2\phi(r) = h\sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} \quad \ldots\ldots\ldots\ldots\ldots \quad (e),$$

$$\text{or } r^2\frac{ds}{dt} = h\frac{ds}{d\theta};$$

$$\text{whence } r^2\frac{d\theta}{dt} = h,$$

the equation for the equable description of areas.

Squaring (e) and attending to (1), we have

$$\frac{r^4}{h^2}(C - 2\int Pdr) = r^2 + \left(\frac{dr}{d\theta}\right)^2,$$

$$\text{or, putting } r = \frac{1}{u},$$

$$\frac{1}{h^2}\left(C + 2\int\frac{P}{u^2}\frac{du}{d\theta}d\theta\right) = u^2 + \left(\frac{du}{d\theta}\right)^2,$$

or, differentiating and dividing by $2\dfrac{du}{d\theta}$,

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2},$$

the general equation to central orbits.

249. Varying Action. If, in § 242, we assume

$$\frac{v^2}{2} = \int (Xdx + Ydy + Zdz) + H = H - V,$$

(with the notation of § 73) it is evident that H will depend on the initial velocity. Supposing that this and the initial and final co-ordinates vary; then, in addition to the already considered variation of the form of the path between its extremities, upon which the unintegrated part of the value of δA depends, we shall have in δA terms depending on the variations of initial and final positions and of initial velocity.

The additional term in $v\delta v$ is δH, and its integral $t\delta H$ is at once obtained. Hence in this more general variation of the conditions we have in the value of δA the following additional terms, depending on the limits only, and therefore to be treated by themselves,

$$\delta A = \left[\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right]$$
$$- \left[\left(\frac{dx}{dt}\right)_0 \delta x_0 + \left(\frac{dy}{dt}\right)_0 \delta y_0 + \left(\frac{dz}{dt}\right)_0 \delta z_0\right] + t\delta H.$$

Hence, if A could be found in terms of x, y, z, x_0, y_0, z_0 and H, we should have at once the integrals of the equations of motion in the form

$$\frac{\delta A}{\delta x} = \frac{dx}{dt}, \quad \frac{\delta A}{\delta x_0} = -\left(\frac{dx}{dt}\right)_0,$$
&c. &c.,

with the farther condition $\frac{\delta A}{\delta H} = t.$

250. A is, of course, a function of x, y, z, x_0, y_0, z_0, and H, and must satisfy the partial differential equations

$$\left(\frac{dA}{dx}\right)^2+\left(\frac{dA}{dy}\right)^2+\left(\frac{dA}{dz}\right)^2=v^2=2(H-V)\dots\dots\dots\dots\dots\dots (1).$$

and $$\left(\frac{dA}{dx_0}\right)^2+\left(\frac{dA}{dy_0}\right)^2+\left(\frac{dA}{dz_0}\right)^2=2(H-V_0)\dots\dots\dots\dots\dots\dots (2).$$

251. The whole circumstances of the motion are thus dependent on the function A, called by Hamilton the *Characteristic Function.* The above is a brief sketch of the foundation of his theory of *Varying Action,* so far as it relates to the motion of a single free particle. The determination of the function A is troublesome, even in very simple cases of motion; but the fact that such a mode of representation is possible is extremely remarkable.

252. More generally, omitting all reference to the initial point, and the equation § 250 (2) which belongs to it, let us consider A simply as a function of x, y, z. *Then any function,* A, *which satisfies* § 250 (1) *possesses the property that*

$$\frac{dA}{dx},\ \frac{dA}{dy},\ \frac{dA}{dz}$$

represent the rectangular components of the velocity of a particle in a motion possible under the action of the given forces.

For, by partial differentiation of (1), we have

$$\frac{d^2x}{dt^2}=X=-\frac{dV}{dx}=\frac{dA}{dx}\frac{d^2A}{dx^2}+\frac{dA}{dy}\frac{d^2A}{dxdy}+\frac{dA}{dz}\frac{d^2A}{dxdz}.$$

But $$\frac{d}{dt}\left(\frac{dA}{dx}\right)=\frac{dx}{dt}\frac{d^2A}{dx^2}+\frac{dy}{dt}\frac{d^2A}{dxdy}+\frac{dz}{dt}\frac{d^2A}{dxdz}.$$

Comparing, we see that

$$\frac{dx}{dt}=\frac{dA}{dx},\ \frac{dy}{dt}=\frac{dA}{dy},\ \frac{dz}{dt}=\frac{dA}{dz},$$

satisfy this and the other two similar pairs of equations.

253. Also, if α, β be constants, which, along with H, are involved in a complete integral of § 250 (1), *the corresponding path, and the time of its description are given by*

$$\frac{dA}{d\alpha}=\alpha_1,\ \frac{dA}{d\beta}=\beta_1,\ \frac{dA}{dH}=t+H_1,$$

where α_1, β_1, H_1 are three additional constants.

For these equations give, by differentiation,

$$\left.\begin{aligned}\frac{d^2A}{dx\,d\alpha}\frac{dx}{dt}+\frac{d^2A}{dy\,d\alpha}\frac{dy}{dt}+\frac{d^2A}{dz\,d\alpha}\frac{dz}{dt}&=0\\ \frac{d^2A}{dx\,d\beta}\frac{dx}{dt}+\frac{d^2A}{dy\,d\beta}\frac{dy}{dt}+\frac{d^2A}{dz\,d\beta}\frac{dz}{dt}&=0\\ \frac{d^2A}{dx\,dH}\frac{dx}{dt}+\frac{d^2A}{dy\,dH}\frac{dy}{dt}+\frac{d^2A}{dz\,dH}\frac{dz}{dt}&=1\end{aligned}\right\}\ldots\quad\ldots(a).$$

But, differentiating § 250 (1), we get

$$\left.\begin{aligned}\frac{d^2A}{d\alpha\,dx}\frac{dA}{dx}+\frac{d^2A}{d\alpha\,dy}\frac{dA}{dy}+\frac{d^2A}{d\alpha\,dz}\frac{dA}{dz}&=0\\ \frac{d^2A}{d\beta\,dx}\frac{dA}{dx}+\frac{d^2A}{d\beta\,dy}\frac{dA}{dy}+\frac{d^2A}{d\beta\,dz}\frac{dA}{dz}&=0\\ \frac{d^2A}{dH\,dx}\frac{dA}{dx}+\frac{d^2A}{dH\,dy}\frac{dA}{dy}+\frac{d^2A}{dH\,dz}\frac{dA}{dz}&=1\end{aligned}\right\}\ldots\ldots\ldots(b).$$

The values of $\frac{dx}{dt}$, &c. in (a) are evidently equal respectively to those of $\frac{dA}{dx}$, &c. in (b). Hence the proposition.

254. *Equiactional surfaces, i. e.* those whose common equation is

$$A = \text{const.} = C,$$

are cut at right angles by the trajectories.

For the direction-cosines of the normal are obviously proportional to $\frac{dA}{dx}, \frac{dA}{dy}, \frac{dA}{dz}$, that is to $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$.

Thus the determination of equiactional surfaces is resolved into the problem of finding the orthogonal trajectories of a set of given curves in space, whenever the conditions of the motion are given. We cannot, in the present work, spare space for much detail on this very curious subject, and therefore give but one other singular property of these surfaces before applying the principle of Varying Action to an important problem.

Let ϖ be the normal distance at any point between the consecutive surfaces

$$A = C, \text{ and } A = C + \delta C.$$

We have evidently

$$\frac{dA}{dx}\delta x + \frac{dA}{dy}\delta y + \frac{dA}{dz}\delta z = \delta C,$$

or

$$\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z = \delta C,$$

where δx, δy, δz are the relative co-ordinates of any two contiguous points on the two surfaces. If ρ be the length of the line joining these points, θ its inclination to the normal (i.e. the line of motion) this may evidently be written

$$v\rho \cos\theta = v\varpi = \delta C,$$

since $\rho \cos\theta$ is the normal distance between the surfaces.

Thus, *the distance between consecutive equiactional surfaces is, at any point, inversely as the velocity in the corresponding path.*

This may be seen at once as follows; the element of the action is $v\delta s$ (where δs, being an element of the path, is the normal distance between the surfaces) and must therefore be equal to δC.

255. *To deduce, from the principle of Varying Action, the form and mode of description of a planet's orbit.*

In this case it is obvious that the force of gravity $\left(-\frac{\mu}{r^2}\right)$ is equal to $-\frac{dV}{dr}$. Hence the right hand member of § 250 (1) may be written $2\left(H + \frac{\mu}{r}\right)$.

Let us take the plane of xy as that of the orbit, then the equation § 250 (1) becomes

$$v^2 = \left(\frac{dA}{dx}\right)^2 + \left(\frac{dA}{dy}\right)^2 = 2\left(H + \frac{\mu}{r}\right) \text{..............} (1).$$

It is not difficult to obtain a satisfactory solution of this equation; but the operation is very much simplified by the use of polar co-ordinates. With this, (1) becomes

$$\left(\frac{dA}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dA}{d\theta}\right)^2 = 2\left(H + \frac{\mu}{r}\right) \quad \ldots\ldots\ldots\ldots\ldots (2),$$

which is obviously satisfied by

$$\left.\begin{aligned} \frac{dA}{d\theta} &= \text{constant} = \alpha \\ \left(\frac{dA}{dr}\right)^2 &= 2\left(H + \frac{\mu}{r}\right) - \frac{\alpha^2}{r^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (3).$$

Hence

$$A = \alpha\theta + \int dr \sqrt{2\left(H + \frac{\mu}{r}\right) - \frac{\alpha^2}{r^2}} \quad \ldots\ldots\ldots\ldots (4).$$

The final integrals are therefore, by § 253,

$$\frac{dA}{d\alpha} = \mathfrak{A} = \theta - \alpha \int \frac{dr}{r^2 \sqrt{2\left(H + \frac{\mu}{r}\right) - \frac{\alpha^2}{r^2}}} \quad \ldots\ldots\ldots\ldots (5),$$

and

$$\frac{dA}{dH} = t + \epsilon = \int \frac{dr}{\sqrt{2\left(H + \frac{\mu}{r}\right) - \frac{\alpha^2}{r^2}}} \quad \ldots\ldots\ldots\ldots\ldots (6).$$

These equations contain the complete solution of the problem, for they involve four constants, $\mathfrak{A}$, α, H, ϵ. (5) gives the equation of the orbit, and (6) the time in terms of the radius-vector.

256. To illustrate the subject farther, we will deduce the ordinary results of Chaps. V. and VI. from these formulæ. Thus, let θ_0, r_0 denote the polar co-ordinates of any fixed point in the path, from which the action is to be reckoned. We have, by (4),

$$A = \alpha(\theta - \theta_0) + \int_{r_0}^{r} dr \sqrt{2\left(\frac{\mu}{r} + H\right) - \frac{\alpha^2}{r^2}}$$

$$= \int_{r_0}^{r} \frac{2\left(\frac{\mu}{r} + H\right) dr}{\sqrt{2\left(\frac{\mu}{r} + H\right) - \frac{\alpha^2}{r^2}}} \quad \text{..................} (7),$$

because, by (5), $\theta - \theta_0 = \int_{r_0}^{r} \frac{\alpha\, dr}{r^2\sqrt{2\left(\frac{\mu}{r} + H\right) - \frac{\alpha^2}{r^2}}}$.

To integrate (7), remark that (§ 140) $\frac{v^2}{2} < \frac{\mu}{r}$ in an elliptic orbit, and that thus H is negative by § 255 (1).

Put
$$\frac{\mu}{H} = -2a,$$

$$\frac{\alpha^2}{\mu a} = 1 - e^2,$$

$$\text{and } r = a(1 - e\cos\phi),$$

and (7) becomes, after substitution,

$$A = \sqrt{\mu a}\int_{\phi_0}^{\phi}(1 + e\cos\phi)\, d\phi,$$

which is immediately integrable.

It is obvious from § 153 that ϕ represents the excentric anomaly. Measured from the perihelion we have evidently

$$A = \sqrt{\mu a}\,(\phi + e\sin\phi).$$

257. By (6) we have $t = \int_{r_0}^{r} \frac{dr}{\sqrt{2\left(\frac{\mu}{r} + H\right) - \frac{\alpha^2}{r^2}}}$.

By employing the same substitutions as in last section, it is easy to bring this expression into the form

$$t = \sqrt{\frac{a^3}{\mu}} \int_{\phi_0}^{\phi} (1 - e\cos\phi)\, d\phi$$

$$= \sqrt{\frac{a^3}{\mu}} [\phi - e \sin\phi],$$

the formula of §§ 152, 154.

258. By the process of § 152 we see that while $\phi - e\sin\phi$ is proportional to the area described about the center of force, and therefore proportional to the time; $\phi + e\sin\phi$ is proportional to the area described about the other focus, and is, by § 256, proportional to the action. Thus *the time is measured by the area described about one focus, and the action by that about the other.*

An easy verification of this curious result is as follows. With the usual notation we have

$$dA = vds,$$

$$= \frac{h}{p} ds. \quad § 22.$$

But in the ellipse or hyperbola, p' being the perpendicular from the second focus,

$$pp' = \pm b^2.$$

Hence $$dA = \pm \frac{h}{b^2} p' ds,$$

which expresses the result sought.

It is easy to extend this to a parabolic orbit, for which, indeed, the theorem is even more simple.

259. When a particle moves in any curve, it has been shewn (§§ 16, 17), that the acceleration along the radius of absolute curvature of the path is $\frac{v^2}{\rho}$; that is, a force $\frac{mv^2}{\rho}$ is required to deflect the particle from the tangent, which is the path it would take if left to itself.

From this, or by the formula in § 135, we see that if a particle revolve at distance r, with angular velocity ω, about a point, a force $mr\omega^2$ to that point is requisite to maintain the distance r unaltered. This tendency to move in the tangent, which arises from the inertia of matter, was formerly supposed to be due to a force, called *Centrifugal Force*, generated in the particle by its rotation about the point.

We have seen that when the motion of a particle in any path is referred to polar co-ordinates in a plane, the acceleration along the radius vector is

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2. \quad § 15.$$

Now the velocity along r is $\frac{dr}{dt}$, and that perpendicular to it $r\frac{d\theta}{dt}$; hence the first term of the above is the acceleration of the velocity along the radius vector, and the other is the so-called centrifugal force due to a velocity $r\frac{d\theta}{dt}$ in a circle of radius r. The idea of this so-called force is useful, as we have already seen (§ 208), in enabling us to form the equations of motion of a particle in particular cases.

260. *Given the path of a particle, and the manner of its description, to find the requisite forces.*

If X, Y, Z be the required forces for unit of mass, we must have

$$X = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

$$= \frac{ds}{dt}\frac{d}{ds}\left(\frac{ds}{dt}\frac{dx}{ds}\right) = v\frac{d}{ds}\left(v\frac{dx}{ds}\right),$$

with similar expressions for Y and Z. But as the path is given, and the manner of its description, that is v in terms of the co-ordinates, the value of the above expressions is completely known.

Instead of having the velocity at any point, we might have had other conditions, such for instance as that the resultant force is to be in a given direction, &c., but these, like the above, present no difficulty.

261. *A particle moves in a plane, under the action of a central force directed to a point which moves in a given manner in the plane: to find the motion.*

Let x, y, ξ, η be the co-ordinates of the particle and point, at time t. ξ and η are given functions of t. Also let $P=f(r)$ be the central force at distance r. Then

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= -P\frac{x-\xi}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\\ \frac{d^2y}{dt^2} &= -P\frac{y-\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\end{aligned}\right\}\quad\text{..............}\quad(1),$$

are the equations of motion.

The equations of *relative* motion are, of course,

$$\left.\begin{aligned}\frac{d^2(x-\xi)}{dt^2} &= -P\frac{x-\xi}{\sqrt{(x-\xi)^2+(y-\eta)^2}}-\frac{d^2\xi}{dt^2}\\ \frac{d^2(y-\eta)}{dt^2} &= -P\frac{y-\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}}-\frac{d^2\eta}{dt^2}\end{aligned}\right\}\quad\text{......}\quad(2),$$

or, putting ξ_1, η_1, for the relative co-ordinates,

$$\left.\begin{aligned}\frac{d^2\xi_1}{dt^2} &= -P\frac{\xi_1}{\sqrt{\xi_1^2+\eta_1^2}}-\frac{d^2\xi}{dt^2}\\ \frac{d^2\eta_1}{dt^2} &= -P\frac{\eta_1}{\sqrt{\xi_1^2+\eta_1^2}}-\frac{d^2\eta}{dt^2}\end{aligned}\right\}\quad\text{.............}\quad(3).$$

These equations illustrate, in a particular case, the general theorem of § 24; as they contain, in addition to the terms due to the attraction of the fixed center, the two known quantities $-\frac{d^2\xi}{dt^2}$ and $-\frac{d^2\eta}{dt^2}$, the components of acceleration of the center *reversed*.

262. Ex. *Let the central force vary directly as the distance.*

Here $P=\mu\sqrt{\xi_1^2+\eta_1^2}$, and equations (3) of last section become

$$\left.\begin{aligned}\frac{d^2\xi_1}{dt^2}&=-\mu\xi_1-\frac{d^2\xi}{dt^2}\\ \frac{d^2\eta_1}{dt^2}&=-\mu\eta_1-\frac{d^2\eta}{dt^2}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

which are easily integrated, in the form

$$\left.\begin{aligned}\xi_1&=A\cos(\sqrt{\mu}t+B)-\frac{\left(\frac{d}{dt}\right)^2}{\left(\frac{d}{dt}\right)^2+\mu}\xi\\ \eta_1&=C\cos(\sqrt{\mu}t+D)-\frac{\left(\frac{d}{dt}\right)^2}{\left(\frac{d}{dt}\right)^2+\mu}\eta\end{aligned}\right\}\ldots\ldots\ldots(5);$$

for particular values of ξ and η in terms of t.

As a particular case, suppose the center of force to move with uniform acceleration, α, parallel to a given direction, which may be taken as the axis of y. The center will in general (Chap. IV.) describe a parabola, and the relative motion of the particle will be the same as in § 125, the center of the ellipse or hyperbola being not at the center of force but at a distance $\frac{\alpha}{\mu}$ from it in a line parallel to the axis of y.

263. *If the radius vector of a curve in space be at each instant parallel to the direction, and equal to the magnitude, of the velocity of a particle moving in any path; the curve is called the hodograph corresponding to the path* (§ 18).

The hodograph is evidently a plane curve if the path is so.

Let x, y, z be the co-ordinates of a point in the path, ξ, η, ζ those of the corresponding point of the hodograph; then evidently by the definition,

$$\left.\begin{aligned}\frac{dx}{dt}&=\xi\\ \frac{dy}{dt}&=\eta\\ \frac{dz}{dt}&=\zeta\end{aligned}\right\}.$$

Hence, if σ be the arc of the hodograph,

$$\frac{d\sigma}{dt}=\sqrt{\left\{\left(\frac{d\xi}{dt}\right)^2+\left(\frac{d\eta}{dt}\right)^2+\left(\frac{d\zeta}{dt}\right)^2\right\}}$$

$$=\sqrt{\left\{\left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2+\left(\frac{d^2z}{dt^2}\right)^2\right\}},$$

and the direction cosines of $\delta\sigma$ are proportional to

$$\frac{d^2x}{dt^2},\ \frac{d^2y}{dt^2},\ \frac{d^2z}{dt^2}.$$

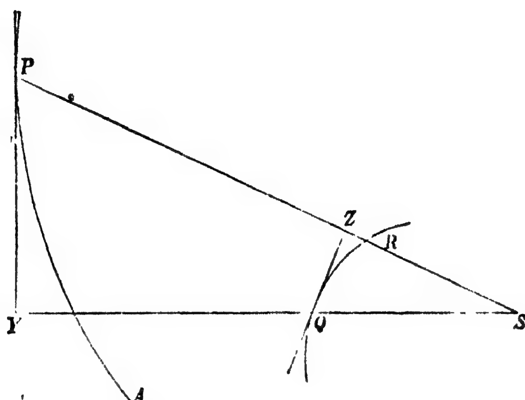
Hence we see as in § 18 that

The tangent to the hodograph at any instant is parallel to the resultant force acting on the particle at the corresponding point of its path, and the velocity in it is equal to the acceleration of the particle.

264. The most important case of the hodograph being that corresponding to an orbit about a single center of force, we may deduce the above properties for that case in a somewhat different manner.

Let P be any point in PA, an arc of an orbit described about a center of force S. Draw SY perpendicular to the tangent at P, and take $SQ.SY=h$, then evidently SQ is equal to the velocity at P, and perpendicular to it in direction. Hence the locus of Q is the hodograph turned in its own plane through 90°.

But we see that it is the polar reciprocal of PA with regard to a circle whose center is S and radius $= \sqrt{h}$. Hence,



by geometry, the tangent at Q is perpendicular to SP. This evidently corresponds to the first of the two general properties of the hodograph given in last section.

Let $r, \theta, p, s, r', \theta', p', s'$ represent the usual quantities for corresponding points of the two curves; then if ρ be the radius of curvature at Q, we have by the condition that QZ is perpendicular to SP,

$$\delta s' = \rho\delta\theta = r'\frac{dr'}{dp'}\delta\theta$$

$$= \frac{h}{p}\frac{d\frac{1}{p}}{d\frac{1}{r}}\delta\theta = \frac{h}{p^3}\frac{dp}{dr}r^2\delta\theta$$

$$= \frac{h^2}{p^3}\frac{dp}{dr}\delta t = P\delta t, \text{ (§ 130)},$$

which proves the second property.

265. *To find the mode of description of a given hodograph that it may correspond to a central orbit.*

Here, $$\rho\delta\theta = \delta s' = \frac{r'^2\delta\theta'}{p'};$$

wherefore $$\rho r^2 \frac{d\theta}{dt} = \frac{h^2 r'^2}{p'^3} \frac{d\theta'}{dt},$$

or $$h\frac{d\theta'}{dt} = \frac{\rho p'^3}{r'^2} = \frac{\rho r'^4}{\left(\frac{ds'}{d\theta'}\right)^3}$$

$$= \frac{r'^4}{r'^2 + 2\left(\frac{dr'}{d\theta'}\right)^2 - r'\frac{d^2r'}{d\theta'^2}}$$

$$= \frac{1}{\left(\frac{d^2u'}{d\theta'^2} + u'\right)u'} \quad \left(\text{where } u' = \frac{1}{r'}\right),$$

which gives the required angular velocity at any point of the hodograph, in terms of the co-ordinates of that point.

266. When the central force is inversely as the square of the distance, we have by § 264 for the arc of the hodograph,

$$\delta s' = \frac{\mu}{r^2}\,\delta t = \frac{\mu}{h}\,\delta\theta;$$

or $$\rho = \frac{\mu}{h}.$$

Hence for all conic sections described about the focus the hodograph is a circle.

This might have been shewn in another way, thus. In the fig. (§ 264) if PA be a portion of an ellipse or hyperbola of which S is the focus, the locus of Y is the auxiliary circle. Hence evidently the locus of Q is a circle. If PA be a portion of a parabola of which S is the focus, the locus of Y is a straight line, and therefore that of Q is a circle passing through S.

Hence generally, the hodograph for any orbit about a center of force whose intensity is inversely as the square of

the distance, is a circle; about an internal point for an ellipse, an external point for a hyperbola, and about a point in the circumference for a parabola.

A purely analytical proof of the same theorem is easily given. If x, y be the co-ordinates of the planet, ξ, η those of a point in the hodograph, then

$$\xi = \frac{dx}{dt}, \quad \eta = \frac{dy}{dt}.$$

The equations of motion are

$$\frac{d^2x}{dt^2} = \frac{\mu x}{r^3},$$

$$\frac{d^2y}{dt^2} = \frac{\mu y}{r^3}.$$

Hence, as usual,

$$x\frac{dy}{dt} - y\frac{dx}{dt} = h \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and therefore

$$\frac{d^2x}{dt^2} = \frac{\mu}{h}\,\frac{x^2\dfrac{dy}{dt} - xy\dfrac{dx}{dt}}{r^3} = \frac{\mu}{h}\,\frac{(x^2+y^2)\dfrac{dy}{dt} - y\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right)}{r^3},$$

which gives, by integration,

$$\left.\begin{aligned} \frac{dx}{dt} + A = \xi + A = \frac{\mu y}{hr}. \\ \text{Similarly} \quad \frac{dy}{dt} + B = \eta + B = -\frac{\mu x}{hr}, \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots (2),$$

and thence

$$(\xi + A)^2 + (\eta + B)^2 = \frac{\mu^2}{h^2},$$

proving that the hodograph is a circle.

Also, by eliminating $\frac{dx}{dt}, \frac{dy}{dt}$ among the three equations (1), (2), we get for the equation to the orbit

$$-h + Ay - Bx = \frac{\mu}{h} r,$$

which gives the focus and directrix property at once.

267. The law of diffusion of heat and light from a calorific and luminous body is that of the inverse square of the distance. Hence an arc of the hodograph of a planet's orbit, which arc we have already seen to represent the entire acceleration due to the central force, represents also the entire amount of light or heat derived from the Sun during the passage through the corresponding arc of its orbit.

Ex. *Compare the amounts of light and heat received throughout their orbits by the Earth moving in a circle, and a comet moving in a parabola at the same perihelion distance.*

The hodographs are both circles, one about its center, the other about a point in its circumference; but the diameter of the latter is $\sqrt{2}$ times the radius of the former, (§ 140).

Hence their circumferences are as $\sqrt{2} : 1$, or the Earth in its orbit receives in a revolution $\sqrt{2}$ times the amount of light and heat, which the comet can receive in its whole path.

It is evident that the path, apparently described by a fixed star in consequence of the *Aberration* of light, is the Hodograph of the Earth's orbit, and is therefore a circle in a plane parallel to the ecliptic.

268. It is evident that that diameter of the circular hodograph which passes through the center of force is divided by the center of force in the same ratio as the axis major of the orbit is divided by the focus; and by (§ 266) its length $= \frac{2\mu}{h}$.

269. Sir W. R. Hamilton enunciates (*Lectures on Quaternions*, p. 614) the following proposition:

If two circular hodographs, having a common chord, which passes through, or tends to, a common center of force, be both cut perpendicularly by a third circle, the times of hodographically describing the intercepted arcs will be equal.

It is evident from (§ 268), that the two orbits are conic sections of the same species, and with equal major axes.

Also, every circle which cuts both hodographs perpendicularly must have its center on the common chord. Let the

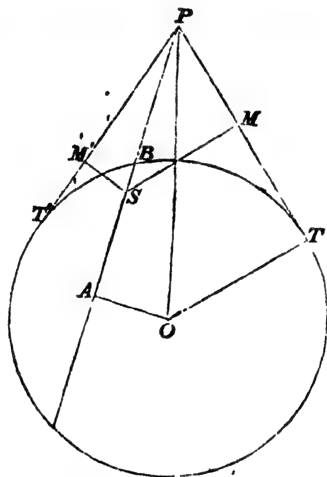


figure represent one of the hodographs, S being the center of force, and ABP the common chord. Take any point P and draw the tangents PT, PT'. We proceed to investigate the difference of the times of hodographically describing TT' and the corresponding arc for a position of P slightly shifted along AP.

Draw OA perpendicular to AP. Let $OT = a$, $AB = b$, $OA = c$, $SP = r$, $SM = \varpi$, $PO = q$, $PA = r'$, and $PT = \tau$. If P be moved through a space δr, the increase of the angle

PSM which is the angle vector in the orbit, is $\frac{\varpi\delta r}{r\tau}$ nearly. But the corresponding radius vector in the orbit is $\frac{h}{\varpi}$ (§ 264) and therefore the time of hodographically describing the small arc at T is

$$\delta t = \frac{1}{h}\frac{h^2}{\varpi^2}\frac{\varpi\delta r}{r\tau} = \frac{\mu\delta r}{r\tau}\frac{1}{\varpi a}.$$

Hence the whole change produced in the time of hodographically describing the arc TT' by shifting P is

$$\frac{\mu\delta r}{r\tau}\left(\frac{1}{a\varpi} + \frac{1}{a\varpi'}\right) = \frac{2\mu r'\delta r}{b^2 r^2 \tau}.$$

[This is easily seen, if we notice that by the figure

$$\left.\begin{matrix}\varpi \\ \varpi'\end{matrix}\right\} = r\sin\left\{\sin^{-1}\frac{a}{q} \pm \sin^{-1}\frac{c}{q}\right\}.]$$

Now this is the same for both hodographs, and, as the arc TT' vanishes for each when P is at B, we have the proposition.

It will readily be seen that this is in substance the same as Lambert's Theorem, (§§ 165, 166).

270. We now take an instance of the determination, from the hodograph and the law of its description, of the curve described and the forces acting.

The hodograph is a circle described with uniform angular velocity about a point in its circumference, find the original path and the circumstances of its description.

Here we have in the hodograph,

$$\rho = a\cos\theta,$$

$$\theta = \omega t;$$

therefore in the path

$$\frac{dx}{dt} = \rho\cos\theta = a\cos^2\omega t,$$

$$\frac{dy}{dt} = \rho \sin\theta = a \cos\omega t \sin\omega t.$$

Integrating and properly adapting the constants, as they affect only the position of the origin,

$$x = \frac{a}{4\omega}(2\omega t + \sin 2\omega t),$$

$$y = \frac{a}{4\omega}(1 - \cos 2\omega t).$$

Now the equations to a cycloid are

$$x = A(\phi + \sin\phi),$$
$$y = A(1 - \cos\phi);$$

hence the path is a cycloid; and, since $2\omega t = \phi$, the direction of motion revolves uniformly. The particle moves under the action of a constant force perpendicular to the base of the cycloidal constraining curve, and the velocity at any point is that due to the distance from the base. The converse is easily proved.

EXAMPLES.

(1) INVESTIGATE the differential equation to the path of a particle in a plane

$$2X = \frac{d}{dx}\left(\frac{Y - X\dfrac{dy}{dx}}{\dfrac{d^2y}{dx^2}}\right).$$

(2) A particle slides down an inverted cycloid from rest at the cusp; shew that the whole acceleration at any instant is g, and that its direction is to the center of the generating circle.

(3) When the whole force on a particle is along the radius vector, shew that the centrifugal force from the pole

$$= \frac{h^2}{r^3}.$$

(4) A particle moves in a plane under the action of any forces, whose resolved parts are P in the radius vector, and T perpendicular to the radius vector. Shew that the chord of curvature through the pole is $\dfrac{2v^2}{P+T\cot\psi}$, where ψ is the exterior angle between the radius vector and tangent.

(5) A catenary is freely described under the action of a force parallel to the axis; shew that the centrifugal force is constant.

(6) A particle, projected from the origin along the axis of y, describes the curve $y^2=4ax$ under the action of a force μy parallel to y, and another parallel to x; shew that

$$v^2=\mu\left(1+\frac{y^2}{4a^2}\right)(c^2+y^2).$$

(7) The curve $y=\phi(x)$ touches the axis of y at the origin, and is described freely by a particle under the action of forces Y parallel to y, and f parallel to x; shew that

$$Y=2f\sqrt{x}\frac{d}{dx}\left\{\sqrt{x}\frac{dy}{dx}\right\},$$

and that if $Y\propto x^n$, n is negative.

(8) A particle describes the hyperbola $x^2=y^2+a^2$, so that

$$\left(\frac{dx}{dt}\right)^2=\left(\frac{dy}{dt}\right)^2-c^2.$$

Find the forces.

(9) The velocity of a particle in a central orbit varies as $\dfrac{1}{r^2}$. Apply the principle of Least Action to find the orbit, and thence the law of force.

(10) Shew that the amount of heat and light received by a planet in one revolution is inversely as the square root of the latus rectum of its orbit.

(11) If P, P' be the central forces for an orbit and its hodograph,

$$PP' = \frac{h'^2}{h^2} rr'.$$

(12) The hodograph is a circle about a point in its circumference, and if θ be the angle which the radius vector makes with the diameter, the angular velocity is given by

$$\frac{d\theta}{dt} = \frac{k}{\sqrt{(\epsilon^{2kt} - 1)}};$$

shew that the path is a cycloid with its vertex upwards, and the velocity at any point, that due to a fall from the tangent at the vertex.

(13) The hodograph for a particle moving in a vertical circle with the velocity due to the depth below the highest point, is

$$r = c \cos \frac{\theta}{2}.$$

(14) When the hodograph is a straight line described uniformly, the path is the trajectory of a projectile in vacuo.

(15) When it is a straight line described with uniform angular velocity about a point, the path is the catenary of uniform strength,

$$\epsilon^{ky} = \sec kx.$$

(16) The hodograph for a circle about a point in the circumference, is a parabola about the focus described with angular velocity proportional to the radius vector.

(17) Determine the motion of a simple pendulum, oscillating in small arcs, when its point of suspension describes, uniformly, a horizontal circle.

Explain the peculiarity of the solution when the time of rotation of the point of suspension is equal to that of a complete oscillation of the pendulum.

(18) Apply the principle of Varying Action to the investigation of the motion of a simple pendulum, slightly disturbed in any manner from its position of equilibrium.

(19) Find the form of the surfaces of equal action for particles projected horizontally from points of a vertical line, the velocity being due to the distance from a given horizontal plane.

(20) Find a central orbit whose form and mode of description correspond with those of the hodograph of another central orbit.

Shew that there is but one law of central force for which this is possible. § 265.

(21) A particle is acted on by a repulsive force tending from a fixed point, and by another force parallel to a fixed line, and when the particle is at a distance r from the fixed point, the magnitudes of these forces are

$$\frac{\mu}{r^2}\left(1-\frac{r}{a}\right) \text{ and } \frac{\mu}{r^2}\left(\frac{r^2}{c^2}+\frac{r}{a}\right),$$

μ, a, c being constants; shew that if the particle be abandoned to the action of the forces at any point at which they are equal to each other, it will proceed to describe a parabola of which the fixed point is the focus.

(22) A particle is acted on by a force the direction of which always meets an infinite straight line AB at right angles, and the intensity of which is inversely proportional to the cube of the distance of the particle from the line. The particle is projected with the velocity from infinity from a point P at a distance a from the nearest point O of the line in a direction perpendicular to OP, and inclined at the angle α to the plane AOP. Prove that the particle is always on the sphere of which O is the center; that it meets every meridian line through AB at the angle α; and that it reaches the line AB in the time $\frac{a^2}{\sqrt{\mu}\cos\alpha}$, μ being the absolute force.

CHAPTER X.

IMPACT.

271. We come next to the consideration of the effects of a class of forces which cannot be treated by the methods just employed. These are called *Impulsive* forces, and are such as arise in cases of collision; lasting for an indefinitely short time, and yet producing finite changes of momentum. Hence, in such questions, *finite* forces need not be considered.

When two balls of glass or ivory impinge on one another, no doubt there goes on a very complicated operation during the brief interval of contact. First, the portions of the surfaces immediately in contact are disfigured and compressed until the molecular forces thus called into action are sufficient to resist farther distortion and compression. At this instant it is evident that the points in contact are moving with the same velocity. But, most substances being endowed with a certain degree of elasticity, the balls tend to recover their spherical form, and an additional pressure is generated; proportional, it is found by experiment, to that exerted during the compression. The coefficient of proportionality is a quantity determinable by experiment, and may be conveniently termed the *Coefficient of Restitution*. It is always less than unity.

The method of treating questions involving forces of this nature will be best explained by taking as an example the case of direct impact of one spherical ball on another; first, when the balls are inelastic. Again, when their coefficient of restitution is given.

And it is evident that in the case of direct impact of spheres we may consider them as mere particles, since everything is symmetrical about the line joining their centers.

272. Suppose that a sphere of mass M, moving with a velocity v, overtakes and impinges on another of mass M',

moving in the same direction with velocity v'; and that at the instant when the mutual compression is completed, the spheres are moving with a common velocity V. If P be the common action between them at any time t during the compression, it must evidently be of the nature of a pressure exerted by each on the other; and we have, if τ be the time during which compression takes place,

$$M(v-V)=\int_0^\tau Pdt=R, \text{ suppose,}$$

$$M'(V-v')=\int_0^\tau Pdt=R;$$

whence $$V=\frac{Mv+M'v'}{M+M'}, \text{ and } R=\frac{MM'}{M+M'}(v-v').$$

From these results we see that the whole momentum after impact is the same as before, and that the common velocity is that of the center of inertia before impact. Had the balls been moving in opposite directions, v' would have been negative, and (taking it positively)

$$V=\frac{Mv-M'v'}{M+M'}, \text{ and } R=\frac{MM'}{M+M'}(v+v').$$

From this it appears that both balls will be reduced to rest if

$$Mv=M'v';$$

that is, if their momenta were originally equal and opposite.

This is the complete solution of the problem if the balls be inelastic, or have no tendency to recover their original form after compression.

273. If the balls be elastic, there will be generated, by their tendency to recover their original forms, an additional action proportional to R.

Let e be the coefficient of restitution, v_1, v_1', the velocities of the balls when finally separated. Then, as before,

$$M(V-v_1)=eR,$$

$$M(v_1'-V)=eR;$$

whence

$$Mv_1 = M\frac{Mv + M'v'}{M+M'} - e\frac{MM'}{M+M'}(v-v'),$$

and

$$v_1 = \frac{(M-eM')v + M'(1+e)v'}{M+M'} = v - \frac{M'}{M+M'}(1+e)(v-v'),$$

with a similar expression for v_1'.

A rather singular result may easily be deduced from the last formula. Suppose $M=M'$, $e=1$, that is, let the balls be of equal mass, and their coefficient of restitution unity (or, in the usual, but most misleading phraseology, "Suppose the balls to be *perfectly elastic*"); then in this case

$$v_1 = v', \text{ and similarly } v_1' = v,$$

or the balls, whatever be their velocities, interchange these, and the motion is the same as if they had passed through one another without exerting any mutual action whatever.

274. The only other case which we can treat in the present work, is that of oblique impact when the balls are perfectly smooth, for in rough and non-spherical balls rotations are generated and the motion of each ball requires to be treated as that of a rigid body.

The simplest case is that of *a particle impinging with given velocity, and in a given direction, on a smooth plane.*

Suppose the plane of the particle's motion to be taken as that of reference; its trace on the given plane as the axis of x, and the point at which the impact takes place, as origin.

The impulsive effect of the plane will evidently be perpendicular to it, since it is smooth. Let this be called R; and let the velocity of the particle be resolved into two v_x, v_y, respectively parallel to the axes. For the first part of the impact,

$$M(v_x - v_x') = 0,$$

$$M(v_y - v_y') = R.$$

But v_y', being the common velocity of the plane and ball, is evidently zero; hence

$$v_x' = v_x, \quad v_y' = 0,$$

or, the velocity parallel to the plane is unchanged, while that perpendicular to it is destroyed. So far for an inelastic ball. If the ball be elastic, let v_x'', v_y'' be the final velocities, then

$$M(v_x' - v_x'') = 0,$$

$$M(v_y' - v_y'') = eR.$$

These equations give

$$v_x'' = v_x' = v_x,$$

shewing that the velocity parallel to the plane is unaffected; and

$$Mv_y'' = -eR = -eMv_y,$$

$$\text{or,} \quad v_y'' = -ev_y,$$

that is, the velocity perpendicular to the plane is reversed in direction, and diminished in the ratio $e : 1$.

If we designate by the name of angle of incidence the inclination of the original direction of the ball's motion to the normal to the plane, and give that of angle of reflexion to the angle made with the same line by the path after impact; then denoting the total velocities before and after impact by V and V'', and these angles by θ, ϕ respectively, we have

$$V \sin\theta = v_x, \quad V'' \sin\phi = v_x'',$$

$$V \cos\theta = v_y, \quad V'' \cos\phi = v_y'';$$

and the previous results give at once

$$\left.\begin{aligned} e\cot\theta &= \cot\phi \\ V'' &= V\frac{\sin\theta}{\sin\phi} \end{aligned}\right\},$$

formulæ sometimes of use.

Of course these results are applicable to cases of impact

on any smooth surface; by making the legitimate hypothesis that the impact, and its consequences on the motion of the ball, would be the same if for the surface its tangent plane at the point of contact were substituted.

275. *Two smooth spheres, moving in given directions and with given velocities, impinge; to determine the impact and the subsequent motion.*

Let the masses of the spheres be M, M'; their velocities before impact v and v', and let the original directions of motion make with the line which joins the centers at the instant of impact, angles α, α'. These angles may easily be calculated from the data, if the radii of the spheres be given.

It is evident that, since the spheres are smooth, the entire impulsive action takes place in the line joining the centers at that instant, and that therefore the future motion of each sphere will be in the plane passing through this line and its original direction of motion.

Let R be the impulse, e the coefficient of restitution; then since the velocities in the line of impact are $v \cos \alpha$ and $v' \cos \alpha'$, we have for their final values v_1, v_1', after restitution, by § 273, the expressions

$$v_1 = v \cos \alpha - \frac{M'}{M+M'}(1+e)(v \cos \alpha - v' \cos \alpha'),$$

$$v_1' = v' \cos \alpha' + \frac{M}{M+M'}(1+e)(v \cos \alpha - v' \cos \alpha'),$$

and the value of R is

$$\frac{MM'}{M+M'}(1+e)(v \cos \alpha - v' \cos \alpha').$$

Hence, the sphere M has finally a velocity v_1 in the line joining the centers, and a velocity $v \sin \alpha$ in a known direction perpendicular to this, namely in the plane through this and its original direction of motion. And similarly for the sphere M'. Thus the impact is completely determined.

276. Recurring to the equations in § 272, we have

$$M(v - V) = R,$$
$$M'(V - v') = R,$$

and, eliminating V,

$$R = \frac{MM'}{M+M'}(v - v') \quad \text{.................} \quad (1).$$

Hence, if e be the coefficient of restitution, v_1, v_1' the final velocities,

$$\left.\begin{aligned} v_1 &= v - \frac{R(1+e)}{M} \\ v_1' &= v' + \frac{R(1+e)}{M'} \end{aligned}\right\} \quad \text{.................} \quad (2).$$

Hence, $Mv_1 + M'v_1' = Mv + M'v'$, whatever be e, or there is no momentum lost. This is, of course, a direct consequence of the Third Law of Motion.

Again, $\quad Mv_1^2 + M'v_1'^2 = Mv^2 + M'v'^2$

$$- 2R(1+e)(v - v') + R^2(1+e)^2\frac{M+M'}{MM'}$$

$$= Mv^2 + M'v'^2 - R^2(1+e)^2\frac{M+M'}{MM'}\left(\frac{2}{1+e} - 1\right).$$

The last term of the right-hand side is therefore twice the kinetic energy apparently destroyed by the impact. When $e = 0$, its value is a maximum, $\frac{MM'}{M+M'}(v - v')^2$. When $e = 1$, its value is zero; that is, in direct impact when the coefficient of restitution is unity no kinetic energy is lost.

The kinetic energy which appears to be destroyed in any of these cases is, as we see from § 73*, only transformed—partly it may be into heat, partly into sonorous vibrations, as in the impact of a hammer on a bell. But, in spite of this, the elasticity may be *perfect*.

Also by (2),

$$v_1' - v_1 = v' - v + R(1+e)\frac{M+M'}{MM'}$$

$$= e(v - v'), \text{ by (1)}.$$

Hence the velocity of separation is e times that of impact. These results may easily be extended to the more general case of § 275.

277. We proceed to some particular problems illustrating this branch of the subject.

To one extremity of a uniform and perfectly flexible chain, lying in a given curve on a smooth horizontal plane, a given impulsive tension is applied in the direction of the tangent at that extremity; it is required to find the impulsive tension at any other point of the chain.

Let this be T at a point of the chain whose co-ordinates are x, y; and let the initial velocities of that point, parallel to the axes, be v_x, v_y; then, μ being the mass of a unit of length of the chain, we have the following equations:

$$\left.\begin{aligned}\frac{d}{ds}\left(T\frac{dx}{ds}\right) &= \mu v_x\\ \frac{d}{ds}\left(T\frac{dy}{ds}\right) &= \mu v_y\end{aligned}\right\}\text{..........................(1)}.$$

The geometrical condition will be determined as follows. The chain being inextensible, the length of an element δs is invariable, therefore the velocities of its two extremities resolved along it must be the same. This gives evidently

$$\frac{dv_x}{ds}\frac{dx}{ds} + \frac{dv_y}{ds}\frac{dy}{ds} = 0 \text{ .... (2)}.$$

From these equations we proceed to eliminate v_x, v_y.

Differentiating (1) with respect to s, we have

$$\mu\frac{dv_x}{ds} = T\frac{d^3x}{ds^3} + 2\frac{dT}{ds}\frac{d^2x}{ds^2} + \frac{d^2T}{ds^2}\frac{dx}{ds},$$

$$\mu \frac{dv_y}{ds} = T\frac{d^3y}{ds^3} + 2\frac{dT}{ds}\frac{d^2y}{ds^2} + \frac{d^2T}{ds^2}\frac{dy}{ds}.$$

Multiplying these by $\frac{dx}{ds}$, $\frac{dy}{ds}$, respectively, and adding, we have by (2),

$$T\left(\frac{dx}{ds}\frac{d^3x}{ds^3} + \frac{dy}{ds}\frac{d^3y}{ds^3}\right) + 2\frac{dT}{ds}\left(\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2}\right)$$

$$+ \frac{d^2T}{ds^2}\left\{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right\} = 0 \ldots\ldots\ldots\ldots (3).$$

But $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$; differentiate, and we get

$$\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} = 0;$$

differentiating again, and transposing, we have,

$$\frac{dx}{ds}\frac{d^3x}{ds^3} + \frac{dy}{ds}\frac{d^3y}{ds^3} = -\left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right\}$$

$$= -\frac{1}{\rho^2},$$

where ρ is the radius of curvature of the element δs.

By means of these transformations (3) takes the final form

$$\frac{d^2T}{ds^2} - \frac{T}{\rho^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

This cannot of course be integrated unless the form of the chain is known, or ρ given in terms of s.

To find the instantaneous direction of motion of any point, we must find v_x and v_y, and their ratio is the tangent of the angle which the required direction makes with the axis of x. These quantities must be found by means of (1) from the value of T given by (4).

278. *Example* I. As a particular example, suppose the chain to form a semicircle of radius a. Then $\rho = a$, and (4) becomes

$$\frac{d^2T}{ds^2} - \frac{T}{a^2} = 0,$$

whose integral is

$$T = A\epsilon^{\frac{s}{a}} + B\epsilon^{-\frac{s}{a}}.$$

To determine the arbitrary constants, observe that when

$$s = 0, \quad T = T_0$$

the original impulse; and when $s = \pi a$, or at the free extremity of the chain, $T = 0$. Thus we have

$$T_0 = A + B,$$
$$0 = A\epsilon^{\pi} + B\epsilon^{-\pi}.$$

These give $A = -\frac{T_0\epsilon^{-\pi}}{\epsilon^{\pi} - \epsilon^{-\pi}}$, $B = \frac{T_0\epsilon^{\pi}}{\epsilon^{\pi} - \epsilon^{-\pi}}$;

and therefore

$$T = \frac{T_0\left\{\epsilon^{\pi - \frac{s}{a}} - \epsilon^{-\left(\pi - \frac{s}{a}\right)}\right\}}{\epsilon^{\pi} - \epsilon^{-\pi}}$$

$$= T_0 \frac{\epsilon^{(\pi - \theta)} - \epsilon^{-(\pi - \theta)}}{\epsilon^{\pi} - \epsilon^{-\pi}} \quad \text{.................. (5)},$$

if $s = a\theta$, that is if we consider the tension at a point whose distance from the tended end subtends an angle θ at the center.

Suppose now that the axis of y is the tangent at the tended end; that of x being the diameter through that point,

$$\text{then } x = a(1 - \cos\theta),$$
$$y = a \sin\theta.$$

These give $$\frac{dx}{ds} = \sin\theta,$$

$$\frac{dy}{ds} = \cos\theta\,;$$

from which, by (1) and (5), $\dfrac{v_y}{v_x} = \dfrac{\dfrac{d}{d\theta}\left[\cos\theta\left\{\epsilon^{\pi-\theta} - \epsilon^{-(\pi-\theta)}\right\}\right]}{\dfrac{d}{d\theta}\left[\sin\theta\left\{\epsilon^{\pi-\theta} - \epsilon^{-(\pi-\theta)}\right\}\right]}$

Differentiating out, and then substituting different values of θ, we get the initial directions of motion of the corresponding points of the chain. Thus, for the tended end, it will easily be seen that, putting $\theta = 0$, we have

$$\frac{v_y}{v_x} = -\frac{\epsilon^{\pi} + \epsilon^{-\pi}}{\epsilon^{\pi} - \epsilon^{-\pi}}.$$

For the free end

$$\frac{v_y}{v_x} = \infty,$$

as we should expect, since there is initially no force on it parallel to the axis of x.

279. *Example* II. Suppose it be required that the tension at each point should be proportional to the distance from the free end of the chain.

Then l being the length, and s as before,

$$T = T_0\left(1 - \frac{s}{l}\right) \text{ by hypothesis;}$$

$$\therefore \frac{d^2T}{ds^2} = 0, \quad \text{or by (4)} \quad \frac{T}{\rho^2} = 0, \quad \text{or } \rho = \infty,$$

that is, the chain must lie in a straight line, as is otherwise evident.

280. *To find the angle which the initial direction of motion of any element makes with the corresponding tangent.*

Generally, let $\tan\phi = \dfrac{v}{v_x} = \dfrac{\dfrac{d}{ds}\left(T\dfrac{dy}{ds}\right)}{\dfrac{d}{ds}\left(T\dfrac{dx}{ds}\right)}$,

$$\text{and} \quad \tan\psi = \frac{dy}{dx} = \frac{\dfrac{dy}{ds}}{\dfrac{dx}{ds}}.$$

Then $(\phi - \psi)$ is the required angle; and

$$\tan(\phi - \psi) = \frac{\dfrac{d}{ds}\left(T\dfrac{dy}{ds}\right)\dfrac{dx}{ds} - \dfrac{d}{ds}\left(T\dfrac{dx}{ds}\right)\dfrac{dy}{ds}}{\dfrac{d}{ds}\left(T\dfrac{dx}{ds}\right)\dfrac{dx}{ds} + \dfrac{d}{ds}\left(T\dfrac{dy}{ds}\right)\dfrac{dy}{ds}}$$

$$= \frac{T\left(\dfrac{dx}{ds}\dfrac{d^2y}{ds^2} - \dfrac{dy}{ds}\dfrac{d^2x}{ds^2}\right)}{\dfrac{dT}{ds}}$$

$$= \frac{T}{\rho\dfrac{dT}{ds}} \quad \text{..............................} (6).$$

Hence again, if the condition be that every element of the chain is to move initially along the chain, $\phi - \psi = 0$. and therefore $\rho = \infty$, or the chain must lie in a straight line.

281. *To find the absolute initial velocity of any element of the chain.*

Squaring and adding equations (1), after performing the differentiations indicated, we have

$$\mu^2 (v_x^2 + v_y^2) = \left(\frac{dT}{ds}\right)^2 \left\{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right\}$$

$$+ 2T\frac{dT}{ds}\left(\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2}\right)$$

$$+ T^2\left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right\}$$

$$= \left(\frac{dT}{ds}\right)^2 + \frac{T^2}{\rho^2}, \quad (\S\ 277).$$

Again, if v_s be the velocity of an element resolved along the corresponding tangent,

$$v_s^2 = (v_x^2 + v_y^2)\cos^2(\phi - \psi)$$

$$= \frac{1}{\mu^2}\left\{\left(\frac{dT}{ds}\right)^2 + \frac{T^2}{\rho^2}\right\} \frac{\left(\frac{dT}{ds}\right)^2}{\left(\frac{dT}{ds}\right)^2 + \frac{T^2}{\rho^2}}$$

$$= \frac{1}{\mu^2}\left(\frac{dT}{ds}\right)^2,$$

which might have been found at once from

$$\mu v_x = \frac{d}{ds}\left(T\frac{dx}{ds}\right),$$

by taking the axis of x parallel to the element δs.

282. These problems might perhaps have been more readily solved by resolving the forces, at any point of the chain, along and perpendicular to the tangent. Calling v_s, v_ρ, the initial velocities in these directions, we have at once

$$\left.\begin{aligned} \delta T &= \mu v_s \delta s \\ \frac{T}{\rho}\delta s &= \mu v_\rho \delta s \end{aligned}\right\},$$

the direction of v_ρ being towards the center of curvature.

The kinematical condition furnished by the inextensibility of the chain is

$$v_\rho = \rho\frac{dv_s}{ds}.$$

From these equations the foregoing results may be easily deduced. And the reader may easily work out for himself the obvious extension of the processes of this section or § 277 to a chain of varying section originally at rest in the form of a curve of double curvature.

283. The only other case we shall consider is that of a continuous series of indefinitely small impacts, whose effect

is comparable with that of a finite force. The obvious method of considering such a problem is to estimate *separately* the changes in the velocity produced by the finite forces, and by the impacts, in the same indefinitely small time δt, and compound these for the actual effect on the motion in that period.

284. *A spherical rain-drop, descending by the action of gravity, receives continually by precipitation of vapor an accession of mass proportional to its surface;* a *being its radius when it begins to descend, and* r *its radius after the interval* t, *shew that its velocity is given by the equation*

$$= \frac{gt}{4}\left(1 + \frac{a}{r} + \frac{a^2}{r^2} + \frac{a^3}{r^3}\right),$$

the resistance of the air being left out of account.

Let e be the thickness of the shell of fluid deposited in unit of time. Then evidently

$$r = a + et \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Also let $\delta v = \delta_1 v + \delta_2 v$ be the increase of velocity in time δt; the first term due to gravity, the second to the impacts.

Evidently, $\delta_1 v = g\delta t$; and if M be the mass at time t, $\delta(Mv) = 0$ is the condition of the impact.

This gives

$$M\delta_2 v = -v\delta M,$$

$$\text{or} \quad \delta_2 v = -v\frac{4\pi r^2 e\delta t}{\frac{4}{3}\pi r^3} = -\frac{3ev\delta t}{r} = -\frac{3ev\delta t}{a + et}.$$

From these we have

$$\frac{dv}{dt} = g - \frac{3ev}{a + et}.$$

Multiplying by $(a+et)^3$, and transferring the last term to the left-hand side of the equation, it gives by inspection

$$(a+et)^3 v = \frac{g}{4e}(a+et)^4 + C.$$

But $$0 = \frac{g}{4e}a^4 + C \text{ by condition.}$$

Hence $$v = \frac{g}{4e}\left\{(a+et) - \frac{a^4}{(a+et)^3}\right\}.$$

Substituting for e from (1),

$$v = \frac{gt}{4(r-a)}\left(r - \frac{a^4}{r^3}\right)$$

$$= \frac{gt}{4}\left(1 + \frac{a}{r} + \frac{a^2}{r^2} + \frac{a^3}{r^3}\right),$$

as required.

To verify this solution, suppose no moisture to be deposited, then $r = a$, and we have $v = gt$ as it ought to be.

285. *One end,* B, *of a uniform heavy chain hangs over a small smooth pully* A, *and the other is coiled up on a table at* C. *If* B *preponderates, determine the motion.*

The moving force due to gravity is the weight of AB minus that of $AC = \mu g(x-a)$ suppose.

Now in an indefinitely small interval δt, this would generate in the portion BAC of the chain an increment of velocity $\delta_1 v$

$$= \frac{\mu g(x-a)}{\mu(x+a)}\delta t. \quad \text{(See Chap. XII.)}$$

But the whole uncoiled chain, being in motion at the commencement of the interval δt with velocity v, lifts up a portion of length $v\delta t$ from the table during that interval. Hence, if $\delta_2 v$ be the change of velocity arising from this impact, we have by the condition that no momentum is lost,

$$\text{i.e. } V' = \frac{MV}{M+M'},$$

$$v + \delta_2 v = \frac{\mu (x+a) v}{\mu (x+a) + \mu v \delta t},$$

$$\text{or } \delta_2 v = -\frac{v^2 \delta t}{x+a},$$

omitting quantities of the second and higher orders.

Hence as $\frac{\delta v}{\delta t} = \frac{\delta_1 v}{\delta t} + \frac{\delta_2 v}{\delta t}$,

proceeding to the limit

$$\frac{dv}{dt} = v\frac{dv}{dx} = \frac{g(x-a) - v^2}{(x+a)};$$

which gives $(x+a)^2 v \frac{dv}{dx} + v^2 (x+a) = g(x^2 - a^2)$

$$\text{or } (x+a)^2 v^2 = (x+a)^2 \left(\frac{dx}{dt}\right)^2 = 2g \int (x^2 - a^2)\, dx,$$

and this determines for any given initial circumstances the velocity at any instant.

EXAMPLES.

(1) If $e = 1$, one ball cannot be reduced to rest by direct impact on another equal ball, unless the latter is at rest.

(2) If two balls for which $e = 1$ impinge directly with equal velocities, their masses must be as 1 : 3 that one may be reduced to rest.

(3) Shew that if two equal balls ($e < 1$) impinge directly with velocities $\frac{1+e}{1-e} V$ and $-V$, the former will be reduced to rest.

(4) Shew that the mass of the ball which must be interposed directly between M at rest, and M' moving with a given velocity V, so that M may acquire the greatest velocity, is

$$\sqrt{(MM')},$$

and that that maximum velocity is $\dfrac{M'V(1+e)^2}{\{\sqrt{M}+\sqrt{M'}\}^2}$.

(5) Suppose $e=1$, and an infinite number of balls to be interposed, shew that the maximum velocity which can thus be given to M, is

$$V\sqrt{\frac{M'}{M}}.$$

[Note that by the result of the preceding question, the masses must form a geometric series, and the above is easily deduced.]

(6) Particles for which $e=1$ slide down radii vectores from the focus of a parabola whose axis is horizontal and plane vertical. After reflexion at the curve they describe their trajectories. What is the locus of the foci?

(7) A impinges on B, shew that A's deviation is greatest when its tangent is $\dfrac{1}{2}\dfrac{1+e}{1-e}\sqrt{\left(\dfrac{1-e}{2}\right)}$.

(8) A particle for which $e=\dfrac{1}{m}$ is projected from a point in a smooth horizontal plane. Find how far it goes before it ceases to rebound. Shew that the times between successive rebounds are in a geometric series whose ratio is $\dfrac{1}{m}$, and the heights above the plane in another whose ratio is $\dfrac{1}{m^2}$.

(9) A particle for which $e=1$ is projected from the foot of an inclined plane in a direction making an angle β with the plane; the plane is inclined at an angle α to the horizon. Shew that if $2\tan\beta=\tan\alpha$, the particle will return after one rebound to the point of projection.

If there be two rebounds before coming back to the point of projection, and the coefficient of restitution be e,

$$\cot \beta = (1 + e + e^2) \tan \alpha.$$

(10) AOB is the vertical diameter of a circle. A particle for which $e = 1$ slides down any chord AC, and is reflected at BC. The locus of the focus of its path is the circle whose diameter is AO.

(11) If the direction in which one ball is moving when it impinges on another equal ball at rest, bisect the angle between their future directions; then that angle is

$$2 \tan^{-1} \sqrt{e}.$$

(12) If $e = \frac{1}{2}$, find the direction in which a ball must be projected against a smooth vertical wall, so as, with the least possible velocity, to return to the point of projection.

(13) ABC is a triangle, a, b, c the points of contact of the inscribed circle with the sides, a being in BC, &c. Shew that if a particle projected from a to b be reflected to c, $Ab = eCb$, and if it return to a,

$$AB = eAC.$$

(14) A number of balls A, B, C, &c. for which e is given, are placed in a line, A is projected with given velocity so as to impinge on B, B then impinges on C, and so on; find the masses of the balls B, C, &c. in order that each of the balls A, B, C, &c. may be reduced to rest by impinging on the next; and find the velocity of the n^{th} ball after its impact with the $(n-1)^{th}$.

(15) A ball is projected in a given direction within a fixed horizontal hoop, so as to go on rebounding from the surface of the hoop; find the limit to which the velocity will approach, and shew that it attains this limit in a finite time, e being less than 1.

(16) A given inelastic mass is let fall from a given height

on one scale of a balance, and two inelastic masses are let fall from different heights on the other scale, so that the three impacts take place simultaneously; find the relations between the masses and heights that the balance may remain permanently at rest.

(17) A particle moving in a parabola about a center of force in the focus, strikes a hard plane at any point of its path. If it describe a parabola after the impact, find the direction of its axis.

(18) OA, OB are rods in the same vertical plane inclined at angles α, β to the horizon. If a particle, for which $e=1$, projected from A, strike B and continue to rebound between A and B, then T being the time of flight, γ the inclination of AB to the horizon,

$$\tan\gamma = \tfrac{1}{2}(\cot\beta - \cot\alpha),$$

$$\text{and } T^2 = \frac{2c}{g}\,\frac{\sin(\alpha+\beta)}{\sqrt{\{4\sin^2\alpha\sin^2\beta + \sin^2(\alpha-\beta)\}}}.$$

(19) Equal particles revolve in opposite directions about the focus in an ellipse of excentricity $\frac{3}{5}$, and impinge at the nearer apse. Find the distances of future impacts, and shew that if p be the original apsidal distance, the particles fall into the center after the time

$$\frac{\pi}{14}\,\frac{(5p)^{\frac{3}{2}}}{\sqrt{(2\mu)}}.$$

(20) Two equal particles, connected by a string which passes freely through the pole, are constrained to move in the same logarithmic spiral. If they be originally at rest and one be projected with given velocity (so as to increase its distance from the pole), determine the impact.

(21) Three balls (supposed indefinitely small), for which $e=1$, are placed at the corners of a triangle. To find the relations among the masses that the sphere A if projected to strike B, may be reflected to C and from C to its original position. The impacts are supposed to take place at each

corner so that the line joining the centers of the spheres is perpendicular to the opposite side of the triangle; α, β, γ, being the angles of the triangle, we find

$$\frac{A}{B}=\frac{\sin\beta}{\sin(\alpha-\gamma)},\quad \frac{A}{C}=\frac{\sin\gamma}{\sin(\beta-\alpha)}.$$

(22) If a rocket, originally of mass M, throw off every unit of time a mass eM with relative velocity V, and if M' be the mass of the case, &c., shew that it cannot rise at once unless $Ve > g$, nor at all unless $\frac{MVe}{M'} > g$. If it do rise at once vertically, shew that its greatest velocity is

$$V\log\frac{M}{M'}-\frac{g}{e}\left(1-\frac{M'}{M}\right),$$

and the greatest height it reaches

$$\frac{V^2}{2g}\left(\log\frac{M}{M'}\right)^2+\frac{V}{e}\left(1-\frac{M'}{M}-\log\frac{M}{M'}\right).$$

(23) If an infinite number of perfectly elastic material points, equally distributed through a hollow sphere, be set in motion each with any velocity, shew that the resulting continuous pressure (referred to a unit of area) on the internal surface is equal to two-thirds of the kinetic energy of the particles divided by the volume of the sphere.

(24) A comet in moving from one given point to another, throws off at every instant small portions of its mass which always bear the same ratio n to the mass which remains. If v be the velocity with which each particle is thrown off, α the inclination of its direction to the radius vector, prove that the period t will be diminished by

$$\frac{3nvt}{fa}\{(\phi'-\phi)\sqrt{(ap)}\sin\alpha-(r'-r)\cos\alpha\},$$

ϕ and ϕ' being the excentric anomalies, r and r' the focal distances at the given points, a the mean distance, $2p$ the latus rectum, and f the force at distance a.

CHAPTER XI.

DISTURBED MOTION.

286. In the investigation of the motion of a particle subjected to the action of disturbing forces, we may, when the latter are very small in comparison with the forces under which the undisturbed orbit is described, suppose that at any instant the actual orbit is of the same *nature* as the undisturbed, but that its magnitude, form and position are slightly different. By this means the consideration of motion in an orbit whose equation cannot be found, or if found would be of extreme complexity, is reduced to the cases considered in the foregoing Chapters, the only additional process being the determination of the changes of the elements or *Parameters* of the orbit, due to the disturbing forces; these parameters being thus made explicit functions of the time. The principal use of this method is in the planetary theory, and there the elements of the *Instantaneous Orbit* cannot be determined but by approximation, which this method affords us the best means of effecting.

It is not necessary that the *orbit* should be changed by the disturbing force, in order that the method of parameters be applicable; suppose for instance a particle be constrained to move on a given curve; the extent of its oscillations, and the velocity with which it reaches a particular point in the path, for instance, are parameters, and in terms of such the motion may be expressed. This method is therefore applicable to any case of free or constrained motion, always supposing the disturbing forces to be small; the only difference being that in constrained motion there are fewer parameters of which to find the variation.

287. The general principle of the method may be explained as follows.

Let
$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= X + X' \\ \frac{d^2y}{dt^2} &= Y + Y' \\ \frac{d^2z}{dt^2} &= Z + Z' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

be the equations of motion of a particle whose co-ordinates at time t are x, y, z; and suppose, farther, that X', Y', Z' are the sums of the resolved parts of the disturbing forces parallel to the axes. Had there been no disturbing forces, we should have had the equations of motion

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= X \\ \frac{d^2y}{dt^2} &= Y \\ \frac{d^2z}{dt^2} &= Z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now let the solution of equations (2) be

$$\left.\begin{aligned} x &= \phi\,(t, a_1, a_2, \ldots a_6) \\ y &= \chi\,(t, a_1, a_2, \ldots a_6) \\ z &= \psi\,(t, a_1, a_2, \ldots a_6) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

involving six arbitrary constants; and the forms of the functions ϕ, χ, ψ, being known.

Let us remark in passing, that, if the motion were constrained, we should have in addition one or two relations between x, y and z, leading to others between their differential coefficients, so that the *number* of arbitrary constants would be reduced.

Now suppose the solutions of (1) to be of the same form as the expressions in (3), $a_1, a_2, \ldots a_6$ being no longer constants but functions of t to be determined.

The fact of (3) having to satisfy (1), gives us the three following unavoidable equations for the determination of these parameters,

$$\left.\begin{aligned}\frac{d^2\phi}{dt^2} &= X + X' \\ \frac{d^2\chi}{dt^2} &= Y + Y' \\ \frac{d^2\psi}{dt^2} &= Z + Z'\end{aligned}\right\} \text{......................} \quad (4).$$

288. We are at liberty then to make any three additional hypotheses regarding them that we please. The most convenient are those afforded by the condition that not only the expressions for the co-ordinates of the particle, but also the expressions for the resolved parts of the velocity parallel to the co-ordinate axes, should be of the same *form* in the disturbed as in the undisturbed orbit.

Now in the disturbed orbit,

$$\frac{dx}{dt} = \left(\frac{d\phi}{dt}\right) + \left(\frac{d\phi}{da_1}\right)\frac{da_1}{dt} + \left(\frac{d\phi}{da_2}\right)\frac{da_2}{dt} + \dots + \left(\frac{d\phi}{da_6}\right)\frac{da_6}{dt},$$

with similar expressions for $\frac{dy}{dt}$ and $\frac{dz}{dt}$, the brackets being used to express *partial* differentiation.

But in the undisturbed orbit

$$\frac{dx}{dt} = \left(\frac{d\phi}{dt}\right);$$

and similarly for $\frac{dy}{dt}$ and $\frac{dz}{dt}$.

From this we have the three additional relations between a_1, a_2, a_3, a_4, a_5, a_6 and t,

$$\left.\begin{aligned}\left(\frac{d\phi}{da_1}\right)\frac{da_1}{dt} + \left(\frac{d\phi}{da_2}\right)\frac{da_2}{dt} + \dots + \left(\frac{d\phi}{da_6}\right)\frac{da_6}{dt} = 0 \\ \left(\frac{d\chi}{da_1}\right)\frac{da_1}{dt} + \left(\frac{d\chi}{da_2}\right)\frac{da_2}{dt} + \dots + \left(\frac{d\chi}{da_6}\right)\frac{da_6}{dt} = 0 \\ \left(\frac{d\psi}{da_1}\right)\frac{da_1}{dt} + \left(\frac{d\psi}{da_2}\right)\frac{da_2}{dt} + \dots + \left(\frac{d\psi}{da_6}\right)\frac{da_6}{dt} = 0\end{aligned}\right\} \text{.......} \quad (5).$$

And taking these into account as well as equations (2), (3), equations (4) become

$$\left.\begin{aligned}
\left(\frac{d\left(\frac{d\phi}{dt}\right)}{da_1}\right)\frac{da_1}{dt}+\left(\frac{d\left(\frac{d\phi}{dt}\right)}{da_2}\right)\frac{da_2}{dt}+\ldots+\left(\frac{d\left(\frac{d\phi}{dt}\right)}{da_6}\right)\frac{da_6}{dt}&=X' \\
\left(\frac{d\left(\frac{d\chi}{dt}\right)}{da_1}\right)\frac{da_1}{dt}+\left(\frac{d\left(\frac{d\chi}{dt}\right)}{da_2}\right)\frac{da_2}{dt}+\ldots+\left(\frac{d\left(\frac{d\chi}{dt}\right)}{da_6}\right)\frac{da_6}{dt}&=Y' \\
\left(\frac{d\left(\frac{d\psi}{dt}\right)}{da_1}\right)\frac{da_1}{dt}+\left(\frac{d\left(\frac{d\psi}{dt}\right)}{da_2}\right)\frac{da_2}{dt}+\ldots+\left(\frac{d\left(\frac{d\psi}{dt}\right)}{da_6}\right)\frac{da_6}{dt}&=Z'
\end{aligned}\right\}\ldots (6).$$

Equations (5) and (6) suffice to determine the six parameters in terms of t; and it may be remarked that, should any of these quantities themselves appear in the coefficients in these differential equations, they may be treated as constants, since their variations may be neglected for a short time at any period of the motion, on account of the smallness of the disturbing forces. This will evidently amount analytically to neglecting higher powers of the disturbing forces than the first.

289. Supposing then the parameters found as functions of t, if in equations (3) we substitute these values; the result of the subsequent elimination of t will evidently be the two equations to the orbit *actually* described by the particle.

But if t be eliminated considering $a_1, a_2, \ldots a_6$ constant, and *then* their values as functions of t be substituted; the two resulting equations will, for any particular value t' given to t, represent a curve which evidently coincides with the actual path at the time t', and which *would* be from that instant the actual path if the disturbing forces were then to cease. This is called the *instantaneous orbit* at the time $t=t'$, and its form and position must evidently undergo a slow change due to the disturbing force.

290. If the constants be fewer than six, as in the case of constrained motion, so many equations as those just indicated

will not be necessary; but the preceding sketch will enable us to apply the method to any such case. Thus if the particle be constrained to move on a given surface, taking its equation along with equations (3), there will evidently be two of the six constants at once determined completely in terms of the others. And if the motion be on a given curve, four of the constants will be got rid of. In that case, taking tangential resolutions we have, s being the length of the arc described at time t, S the tangential acceleration, S' the tangential disturbing force,

$$\frac{d^2s}{dt^2} = S + S' \qquad (7).$$

Now if

$$s = \phi(a, \alpha, t) \qquad (8)$$

be the solution of

$$\frac{d^2s}{dt^2} = S,$$

we shall get the values of a, α in terms of t, in order that (8) may satisfy (7), and also that the expression for the velocity in the orbit may be the same in both cases, by solving the equations

$$\left(\frac{d\phi}{d\alpha}\right)\frac{d\alpha}{dt} + \left(\frac{d\phi}{da}\right)\frac{da}{dt} = 0,$$

$$\left(\frac{d\left(\frac{d\phi}{dt}\right)}{d\alpha}\right)\frac{d\alpha}{dt} + \left(\frac{d\left(\frac{d\phi}{dt}\right)}{da}\right)\frac{da}{dt} = S',$$

which evidently correspond to those of groups (5) and (6) respectively.

291. *To determine the effect of a small disturbing force on a simple cycloidal pendulum.*

If s be the arcual distance of the particle from the lowest point at time t we have, (§ 173), putting

$$\frac{g}{4a} = n^2,$$

$$\frac{d^2s}{dt^2} + n^2 s = 0,$$

as the equation of undisturbed motion. Its solution is

$$s = a \sin(nt + b),$$

where a and b are arbitrary constant quantities depending on the length of the arc of vibration and the time of passing the lowest point.

The velocity at time t is $\frac{ds}{dt} = na \cos(nt + b)$.

We shall now suppose that f is a small tangential disturbing force: the equation of motion is

$$\frac{d^2s}{dt^2} + n^2 s = f.$$

The solution of this equation we assume to be

$$s = a \sin(nt + b),$$

a and b being considered unknown functions of t, which it is our business now to determine.

Taking as a condition, that the form of the expression for the velocity is to be still the same; since we have

$$\frac{ds}{dt} = na \cos(nt + b) + \frac{da}{dt} \sin(nt + b) + a \cos(nt + b) \frac{db}{dt},$$

$$\text{we have } \frac{da}{dt} \sin(nt + b) + a \cos(nt + b) \frac{db}{dt} = 0,$$

which is the *assumed* relation between a and b.

$$\text{Again, since } \frac{ds}{dt} = na \cos(nt + b),$$

$$\frac{d^2s}{dt^2} = -n^2 a \sin(nt+b) + n \frac{da}{dt} \cos(nt+b) - na \sin(nt+b) \frac{db}{dt}.$$

In this substitute for $\frac{d^2s}{dt^2}$ its value from the equation of motion, and we have

$$n\frac{da}{dt}\cos(nt+b) - na\sin(nt+b)\frac{db}{dt} = f,$$

which is the second equation connecting a and b with t.

Eliminating successively $\frac{db}{dt}$ and $\frac{da}{dt}$ from these, we have

$$\frac{da}{dt} = \frac{f}{n}\cos(nt+b), \quad \frac{db}{dt} = -\frac{f}{na}\sin(nt+b).$$

If we could solve these equations exactly we should have the complete determination of the motion. In few cases is this practicable: in all to which we shall have to apply the investigation an approximation is sufficient.

We suppose f to be a very small force. Hence the variable parts of a and b are of the same order of magnitude as f, and may be neglected on the right-hand side of the above equations if we agree to neglect the square and higher powers of f.

In order to find the alteration in the extent of vibration which takes place in one oscillation we must integrate $\frac{f}{n}\cos(nt+b)\,dt$ between the limits of t corresponding to one oscillation; that is, from a value of t which gives $nt+b=\alpha$ to the value of t which gives $nt+b=\pi+\alpha$. Here α may be any quantity: in different cases we shall find it convenient to integrate between different limits.

Hence, *increase of arc of semi-vibration* $= \frac{1}{n}\int f\cos(nt+b)\,dt$ between the above-mentioned limits.

To find the alteration in the time of oscillation, let T, T' be the values of t at two successive arrivals of the pendulum at the lowest point; B, B' the values of b at these times. Then

$$nT + B = m\pi, \quad nT' + B' = (m+1)\pi;$$

$$\therefore n(T'-T) + B' - B = \pi,$$

$$T' - T = \frac{\pi}{n} - \frac{1}{n}(B'-B).$$

Now $B' - B = \int_T^{T'} \frac{db}{dt} dt = -\frac{1}{na}\int_T^{T'} f \sin(nt + b)\, dt$:

therefore the increase of time of oscillation

$$= \frac{1}{n^2 a}\int_T^{T'} f \sin(nt + b)\, dt,$$

and *the proportionate increase of time of oscillation*

$$= \frac{1}{\pi na}\int_T^{T'} f \sin(nt + b)\, dt.$$

If the circumstances are such that we must integrate through two vibrations, then

proportionate increase of time of oscillation

$$= \frac{1}{2\pi na}\int f \sin(nt + b)\, dt.$$

292. These formulæ are convenient when f can be expressed in terms of t. If however f be expressed in terms of s, as is the case particularly in clock escapements, we must modify the formulæ: thus

$$\frac{da}{ds} = \frac{da}{dt}\frac{dt}{ds} = \frac{1}{na\cos(nt + b)}\frac{da}{dt} = \frac{f}{n^2 a},$$

and
$$\frac{db}{ds} = \frac{1}{na\cos(nt + b)}\frac{db}{dt}$$

$$= -\frac{f}{n^2 a^2}\tan(nt + b) = -\frac{f}{n^2 a^2}\frac{s}{\sqrt{(a^2 - s^2)}}.$$

Hence, *increase of arc of semi-vibration* $= \frac{1}{n^2 a}\int_{-s}^{s} f ds$,

proportionate increase of the time of vibration

$$= \frac{1}{\pi n^2 a^2}\int_{-s}^{s}\frac{fsds}{\sqrt{(a^2 - s^2)}}.$$

The limits should strictly be $-s$ and s', where s' differs from s by a quantity which depends upon the change in the arc of vibration: but we may neglect this difference between s and s', since the terms in which they occur are small.

293. *Instead of vibrating in a cycloid, let the pendulum vibrate in a circle.*

Here the force $= g \sin \frac{s}{l} = \frac{gs}{l} - \frac{gs^3}{6l^3}$ nearly;

$$\therefore f = \frac{g}{6l^3} s^3 = \frac{ga^3}{6l^3} \sin^3 (nt + b);$$

therefore proportionate increase in time of vibration

$$= \frac{ga^2}{6\pi nl^3} \int \sin^4 (nt + b) \, dt.$$

Now $\int \sin^4 (nt + b) \, dt = \frac{1}{8} \int \{3 - 4 \cos 2 (nt + b) + \cos 4 (nt + b)\} \, dt$

$$= \frac{1}{8} \left\{3t - \frac{2}{n} \sin 2 (nt + b) + \frac{1}{4n} \sin 4 (nt + b)\right\} + C$$

$$= \frac{3}{8} \frac{\pi}{n}, \text{ from } nt + b = 0 \text{ to } \pi;$$

therefore proportionate increase of time

$$= \frac{ga^2}{16n^2l^3} = \frac{a^2}{16l^2} \text{ since } n^2 = \frac{g}{l}.$$

The increase of arc of vibration

$$= \frac{ga^3}{6nl^3} \int \cos (nt + b) \sin^3 (nt + b) \, dt$$

$$= \frac{ga^3}{24n^2l^3} \sin^4 (nt + b) + C = 0 \text{ between the limits,}$$

as we might easily have foreseen.

294. *Suppose the friction at the point of suspension to be constant.*

It will be convenient to take the integrals during that time in which the friction acts in the same direction: that is, from

the beginning of a vibration to its end, or from $nt+b=-\frac{1}{2}\pi$ to $nt+b=\frac{1}{2}\pi$. Here $f=-c$, since the friction *retards* the motion;

$$\therefore \text{ increase of arc} = -\frac{c}{n}\int \cos(nt+b)\,dt$$

$$= -\frac{c}{n^2}\sin(nt+b) + C = -\frac{2c}{n^2},$$

$$\text{proportionate increase of time} = -\frac{c}{\pi na}\int \sin(nt+b)\,dt$$

$$= \frac{c}{\pi n^2 a}\cos(nt+b) + C = 0, \text{ between the limits.}$$

295. *Suppose the resistance of the air to produce a force varying as the* m^th^ *power of the velocity or* $= \mathrm{k}v^m$, m *being any whole number.*

The velocity in moving from the lowest point

$$= na\cos(nt+b); \quad \therefore f = -kn^m a^m \cos^m(nt+b);$$

therefore increase of arc

$$= -kn^{m-1}a^m \int \cos^{m+1}(nt+b)\,dt \text{ from } nt+b=-\tfrac{1}{2}\pi \text{ to } \tfrac{1}{2}\pi$$

$$= -k\pi n^{m-2}a^m \frac{m(m-2)\ldots\ldots\ldots 1}{(m+1)(m-1)\ldots 2}\ (m \text{ odd})$$

$$= -2kn^{m-2}a^m \frac{m(m-2)\ldots\ldots\ldots 2}{(m+1)(m-1)\ldots 3}\ (m \text{ even}).$$

When $m=2$ (the law usually taken) the decrease of the arc $=\frac{4}{3}ka^2$.

The proportionate increase of time of oscillation

$$= -\frac{k}{\pi}n^{m-1}a^{m-1}\int \cos^m(nt+b)\sin(nt+b)\,dt$$

$$= \frac{kn^{m-2}a^{m-1}}{\pi(m+1)}\cos^{m+1}(nt+b) + C = 0, \text{ between limits,}$$

whether m be a positive integer or fraction.

296. *Suppose the resistance of the air is expressed by any function of the velocity.*

Here $f = -\phi(v)$ for motion in the positive direction: and the increase of the arc of vibration

$$= \frac{1}{n^3 a}\int \phi(v) \frac{\cos(nt+b)}{\sin(nt+b)} dv = \frac{1}{n^3 a}\int \frac{v\phi(v)\,dv}{\sqrt{(n^2a^2 - v^2)}}$$

from $v = 0$ to $v = 0$ again. But it must be observed that from $v = 0$ to $v = na$ (that is, from $s = -a$ to $s = 0$) the radical must be taken with a negative sign, because $\sin(nt+b)$ is then negative. The increase of the arc is consequently

$$= -\frac{1}{n^3 a}\int_0^{na} \frac{v\phi(v)\,dv}{\sqrt{(n^2a^2 - v^2)}} + \frac{1}{n^3 a}\int_{na}^{0} \frac{v\phi(v)\,dv}{\sqrt{(n^2a^2 - v^2)}},$$

$$\text{and therefore decrease} = \frac{2}{n^3 a}\int_0^{na} \frac{v\phi(v)\,dv}{\sqrt{(n^2a^2 - v^2)}}.$$

The proportionate increase of time of vibration

$$= -\frac{1}{\pi n a}\int \phi(v)\sin(nt+b)\,dt = \frac{1}{\pi n^3 a^2}\int \phi(v)\,dv$$

$$= \frac{1}{\pi n^3 a^2}\psi(v) = 0, \text{ from } v = 0 \text{ to } v = 0.$$

Hence a resistance which is constant, or which depends on the velocity, does not alter the time of vibration.

297. *Let the resistance be that produced by a current of air moving in the plane of vibration with a velocity* V *greater than the greatest velocity of the pendulum: and varying as the square of their relative velocity.*

Here $f = \phi(v) = k(V - v)^2$ when the pendulum moves in the direction of the current, which we suppose to be the *positive* direction of s; and $f = \phi(v) = k(V + v)^2$ when it moves in the opposite direction.

By the formula in the last Example, when the pendulum moves in the direction of the current, the arc is increased by

$$k\left(\frac{2V^2}{n^2} - \frac{Va\pi}{n} + \frac{4a^2}{3}\right),$$

and when it returns the arc is diminished by

$$k\left(\frac{2V^2}{n^2}+\frac{Va\pi}{n}+\frac{4a^2}{3}\right).$$

The diminution in two vibrations $=\frac{2kVa\pi}{n}$. The time is unaffected.

298. *Let a force* F *act through a very small space* x *at the distance* c *from the lowest point.*

The increase of the arc $=\frac{1}{n^2a}\int_c^{c+x}Fds=\frac{Fx}{n^2a}$ nearly.

The proportionate increase of the time of vibration

$$=\frac{1}{\pi n^2a^2}\int_c^{c+x}\frac{Fsds}{\sqrt{(a^2-s^2)}}.$$

If the general value of the integral be $\phi(s)$, then the proportionate increase of time

$$=\phi(c+x)-\phi(c)=\phi'(c)x$$

$$=\frac{Fx}{\pi n^2a^2}\frac{c}{\sqrt{(a^2-c^2)}}.$$

If then an impulse be given when the pendulum is at its lowest point, $c=0$ and the time of vibration is unaffected.

299. *To determine the motion of a projectile in a uniform medium, the resistance being as the square of the velocity.*

Here as before (§ 220),

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=-kv^2\frac{dx}{ds}=-k\frac{ds}{dt}\frac{dx}{dt}\\ \frac{d^2y}{dt^2}&=-g'-kv^2\frac{dy}{ds}=-g'-k\frac{ds}{dt}\frac{dy}{dt}\end{aligned}\right\}\text{..........}(1).$$

Now if $k=0$, we have evidently

$$\left.\begin{aligned}x&=a+mt\\ y&=b+nt-\tfrac{1}{2}g't^2\end{aligned}\right\}\text{....................}(2),$$

where a, b are the co-ordinates of the point of projection, and m, n the initial horizontal and vertical velocities.

Now supposing (2) to be the solution of (1), a, b, m and n being now functions of t, we have

$$\left.\begin{aligned}\frac{dx}{dt} &= m,\\ \frac{dy}{dt} &= n - g't,\end{aligned}\right\}$$

in both orbits; therefore

$$\left.\begin{aligned}\frac{da}{dt} + t\frac{dm}{dt} &= 0\\ \frac{db}{dt} + t\frac{dn}{dt} &= 0\end{aligned}\right\} \qquad (3).$$

Also $$\frac{ds}{dt} = \sqrt{\{m^2 + (n - g't)^2\}},$$

and equations (1) become

$$\left.\begin{aligned}\frac{d^2x}{dt^2} = \frac{dm}{dt} &= -km\sqrt{\{m^2 + (n - g't)^2\}},\\ \frac{d^2y}{dt^2} + g' = \frac{dn}{dt} &= -k(n - g't)\sqrt{\{m^2 + (n - g't)^2\}}\end{aligned}\right\} \quad \dots (4);$$

and (3) becomes

$$\left.\begin{aligned}\frac{da}{dt} &= kmt\sqrt{\{m^2 + (n - g't)^2\}}\\ \frac{db}{dt} &= k(n - g't)\,t\sqrt{\{m^2 + (n - g't)^2\}}\end{aligned}\right\} \quad \dots\dots\dots (5).$$

The last four equations suffice to determine a, b, m, n in terms of t, and thence the instantaneous orbit. For a first approximation, we may on account of the remark in § 288 integrate these equations on the supposition that the right-hand side of each is variable only so far as it explicitly involves t.

Now equations (2) show that the latus rectum of the instantaneous parabola is $\frac{2m^2}{g'}$; and as $\frac{dm}{dt}$ is negative, by (4), we see that the latus rectum continually diminishes.

Also by equations (2),

$$\left(x-a-\frac{mn}{g'}\right)^2=-\frac{2m^2}{g'}\left(y-b-\frac{n^2}{2g'}\right).$$

These give for the co-ordinates of the focus of the instantaneous orbit

$$\left.\begin{aligned} x' &= a+\frac{mn}{g'} \\ y' &= b+\frac{n^2-m^2}{2g'} \end{aligned}\right\}.$$

If these expressions be differentiated and $\frac{da}{dt}$, &c. be elimirated by means of (4) and (5), it will be seen that $\frac{dx'}{dt}$ is negative, or the axis of the instantaneous orbit moves backwards, until the particle reaches the vertex; after which it progresses for the rest of the motion; also that $\frac{dy'}{dt}$ is positive if $m > n$, that is, the focus of the instantaneous orbit moves upwards while the direction of motion of the particle makes with the horizon an angle less than 45°, i. e. while the particle is above the latus rectum of the instantaneous orbit.

300. *A particle, moving about a center of force whose intensity is inversely as the square of the distance, is subjected to a small disturbing force in its plane of motion; to investigate the change in the form and position of the orbit.*

Let the disturbing force be resolved into two, ϕ and ψ, one along the radius vector and the other perpendicular to it; the equations of motion are

$$\left.\begin{aligned} \frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2 &= -\frac{\mu}{r^2}+\phi \\ \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) &= \psi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (1).$$

Now the solutions of these equations are

$$\left.\begin{aligned}\frac{1}{r} &= \frac{\mu}{h^2}\{1 + e\cos(\theta - \varpi)\}\\ r^2\frac{d\theta}{dt} &= h\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(2),$$

if we omit the forces ϕ and ψ. When we consider their effect then, the quantities h, e and ϖ must be considered variable.

But in the instantaneous orbit, the velocity and direction of motion are the same as in the actual orbit, and therefore if (2) be differentiated, considering h, e, and ϖ variable, the results for r, $\frac{dr}{d\theta}$, and $\frac{d\theta}{dt}$ must be the same in form as if the disturbing forces had not acted. This will enable us to avoid second differential coefficients of h, e and ϖ; and the substitution of their values for $\frac{dr}{dt}$, $\frac{d^2r}{dt^2}$, and $\frac{d\theta}{dt}$ in (1), will give us altogether three equations for

$$\frac{dh}{dt},\ \frac{de}{dt},\ \frac{d\varpi}{dt}.$$

The expressions for these quantities are complicated and so we do not give them. They will be more easily investigated in particular cases, when ϕ and ψ are given.

In the case of the orbit being an ellipse, $h^2 = \mu a(1 - e^2)$, so that we have by substitution

$$\frac{da}{dt},\ \frac{de}{dt},\ \text{and}\ \frac{d\varpi}{dt}.$$

And the second integral of the second of equations (1) involves ϵ or the epoch, which will also be thus found as a function of t.

301. If we desire the change produced in the form and position of an orbit by a slight change made in the velocity, or direction of motion, &c. at some particular point, we must express separately each of the elements of the orbit in terms of the quantity to be changed; then taking the differentials of both sides, we have the required changes of value.

Thus, we have generally in an elliptic orbit

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right). \quad \S\ 142\ (9).$$

At the extremity of the axis major farthest from the focus this becomes

$$V^2 = \frac{\mu}{a}\frac{1-e}{1+e}.$$

Now if at this point V be made $V + \delta V$, without change of direction, we have the condition that in the new orbit $a(1+e)$ shall have the same value as in the old; since this will still be the apsidal distance.

Hence

$$\delta(V^2) = \delta\left(\frac{\mu}{a}\frac{1-e}{1+e}\right),$$

$$\text{and } \delta\{a(1+e)\} = 0;$$

$$\therefore\ 2V\delta V = -\frac{\mu}{a}\frac{\delta e}{1+e},$$

$$\text{or } \delta e = -2\sqrt{\left\{\frac{a}{\mu}(1-e^2)\right\}}\,\delta V.$$

$$\text{And } \delta a = -\frac{a}{1+e}\delta e$$

$$= 2\sqrt{\left(\frac{a^3}{\mu}\frac{1-e}{1+e}\right)}\,\delta V,$$

which determine the increase of the axis major and diminution of the excentricity, and the same method is applicable to more complicated cases.

Again, in the case of a parabolic orbit, as in Chap. IV., it is easy to see that a change in the magnitude of the velocity shifts the focus in the line joining it with the projectile through a space

$$\frac{V\delta V}{g},$$

raises the directrix through an equal space, and increases the latus-rectum by

$$\frac{4V\delta V}{g}\cos^2\alpha,$$

where α is the inclination of the path to the horizon at the instant of the impact.

If the *direction* of motion only be changed, the directrix is unaltered, the focus moves in a direction perpendicular to the line joining it with the projectile, and the latus rectum is diminished by the quantity

$$-\frac{4V^2}{g}\sin\alpha\cos\alpha\delta\alpha.$$

In the latter case the new orbit again intersects the old, and the tangents to either at the two points of intersection are at right angles to each other; so long as the displacement $\delta\alpha$ is indefinitely small.

These results may easily be extended by geometrical processes, as in Chap. IV., or deduced by differentiation from the analytical results there given.

EXAMPLES.

(1) If a small velocity $n\frac{\mu e}{h}$ be impressed on a planet, in the direction of the radius vector, shew that

$$\delta e = ne\sin(\theta - \varpi),$$
$$\delta\varpi = -n\cos(\theta - \varpi).$$

(2) A satellite moves about a spherical planet in the plane of its equator, in a slightly elliptic orbit. Find the motion of the apse due to an uniform mountain ridge at the equator.

(3) Central force varying as the distance, the velocity of a particle is increased by $\frac{1}{n}$th when it is at the extremity of

one of the equal conjugate diameters of its orbit. Shew that each axis is increased by $\frac{1}{2n}$ th, and that the apse regredes through an angle

$$\frac{1}{n}\frac{ab}{a^2-b^2}.$$

(4) At what point of an elliptic orbit described about the focus, can a small change be made in the direction of motion without altering the position of the apse?

If $\delta\phi$ be this change, shew that (in the supposed case)

$$\delta\phi = \frac{\delta e}{1-e^2}.$$

(5) Shew that in an elliptic orbit about the focus, if the velocity be increased by $\frac{1}{n}$ th when the true anomaly is $\theta-\varpi$; we shall have

$$\delta\varpi = \pm\frac{nr\sin(\theta-\varpi)}{ae},$$

as the particle is moving to or from the nearer apse.

(6) A particle moving about a center of force in the focus, in an ellipse, of small excentricity, receives a small impulse perpendicular to its direction of motion at any instant. Find the effect on the position of the apse.

(7) Again, if at the extremity of the axis minor the velocity be increased by $\frac{1}{n}$ th, and the direction changed so that h remains the same, find the alteration in the form and position of the orbit

$$\delta a = 2\left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}\delta V,$$

$$\delta e = \left(\frac{a}{\mu}\right)^{\frac{1}{2}}\left(\frac{1}{e}-e\right)\delta V,$$

(8) The first term of the central disturbing force on the moon is $-m^2r$, where the central force is $\frac{\mu}{r^2}$; shew that the apsidal angle (the orbit being nearly circular) is

$$\pi\left(1+\frac{3}{2}\frac{m^2}{n^2}\right) \text{ nearly,}$$

where $\frac{2\pi}{n}$ is a mean lunar month.

(9) A particle is moving in a circle about a center of force $\propto$ (Dist.)$^{-2}$. The absolute force of the center increases slowly and uniformly. Determine the approximate elements of the orbit after a given time.

(10) A particle moves in a focal elliptic orbit in a very rare medium whose resistance is as the square of the velocity; determine the effect of the resistance on the periodic time.

(11) A particle is projected along a slightly rough inclined plane; find the approximate path, and the velocity at any point.

(12) A point is describing a circle, the acceleration tending to the center and varying inversely as the square of the distance: if the velocity at any point be increased in the ratio of $\sqrt{3}$ to $\sqrt{2}$, find the eccentricity of the new orbit.



CHAPTER XII.

MOTION OF TWO OR MORE PARTICLES.

302. Having considered in detail the various cases which occur in the motion of a single particle subject to the action of any forces, and whose motion is either free, constrained, or resisted, we proceed to the investigation of some very simple cases in which more particles than one are involved. These will divide themselves naturally into two series; first, when the particles are entirely free, and are subject to their mutual attractions as well as to other common impressed forces: and second, when there are in addition constraining forces; such as when two or more of the particles are connected by inextensible strings, &c. Let us take these in order:—

I. *Free Motion.*

303. An immediate application of the third law of motion shews that if two particles attract each other, they exert each on the other equal and opposite forces.

If then m, m', be the masses of the particles, and the force between two units of matter at distance D be $\phi'(D)$, the common force is

$$m m' \phi'(D).$$

304. *A system of free particles is subject to no forces but the mutual attractions; to investigate the motion of the system.*

Let, at time t, x_n, y_n, z_n be the co-ordinates of the particle whose mass is m_n, and let $\phi'(D)$ be the law of attraction. Let ${}_pr_q$ express the distance between the particles m_p and m_q; then we have for the motion of m_1,

$$m_1 \frac{d^2x_1}{dt^2} = \Sigma \left\{ m_1 m_n \phi'({}_1r_n) \frac{x_n - x_1}{{}_1r_n} \right\} \qquad (1),$$

$$m_1 \frac{d^2y_1}{dt^2} = \Sigma \left\{ m_1 m_n \phi'({}_1r_n) \frac{y_n - y_1}{{}_1r_n} \right\} \qquad (2),$$

$$m_1 \frac{d^2z_1}{dt^2} = \Sigma \left\{ m_1 m_n \phi'({}_1r_n) \frac{z_n - z_1}{{}_1r_n} \right\} \qquad (3),$$

with similar equations for each of the others; the sums being taken throughout the system. Before we can make any attempt at a solution of these equations, we must know their number, and the laws of attraction between the several pairs of particles. But some general theorems, independent of these data, may easily be obtained: although not nearly so simply as in Chap. II.

305. I. Conservation of Momentum. In the expression for $m_p \frac{d^2x_p}{dt^2}$, we have a term

$$m_p m_q \phi'({}_pr_q) \frac{x_q - x_p}{{}_pr_q},$$

and in $m_q \frac{d^2x_q}{dt^2}$ we have

$$m_q m_p \phi'({}_qr_p) \frac{x_p - x_q}{{}_qr_p}.$$

Hence if we add all the equations of the form (1) together the result will be

$$m_1 \frac{d^2x_1}{dt^2} + m_2 \frac{d^2x_2}{dt^2} + \ldots\ldots = 0;$$

$$\text{or } \Sigma \left(m \frac{d^2x}{dt^2} \right) = 0.$$

Similarly

$$\Sigma \left(m \frac{d^2y}{dt^2} \right) = 0.$$

$$\Sigma \left(m \frac{d^2z}{dt^2} \right) = 0.$$

Now if $\bar{x}$, $\bar{y}$, $\bar{z}$, be at time t the co-ordinates of the center of inertia of all the particles, § 53,

$$\Sigma (m)\,\bar{x} = \Sigma (mx),$$
$$\Sigma (m)\,\bar{y} = \Sigma (my),$$
$$\Sigma (m)\,\bar{z} = \Sigma (mz).$$

And the above equations may be written,

$$\left.\begin{aligned} \Sigma (m)\frac{d^2\bar{x}}{dt^2} &= 0 \\ \Sigma (m)\frac{d^2\bar{y}}{dt^2} &= 0 \\ \Sigma (m)\frac{d^2\bar{z}}{dt^2} &= 0 \end{aligned}\right\} \text{ or, } \left\{\begin{aligned} \frac{d^2\bar{x}}{dt^2} &= 0, \\ \frac{d^2\bar{y}}{dt^2} &= 0, \\ \frac{d^2\bar{z}}{dt^2} &= 0. \end{aligned}\right.$$

Whence

$$\left.\begin{aligned} \frac{d\bar{x}}{dt} &= a \\ \frac{d\bar{y}}{dt} &= b \\ \frac{d\bar{z}}{dt} &= c \end{aligned}\right\}.$$

These equations shew that the velocity of the center of inertia parallel to each of the co-ordinate axes remains invariable during the motion, that is, that *the center of inertia of the system remains at rest, or moves uniformly in a straight line.* See § 67.

The values of a, b, c, may thus be determined,

$$a = \frac{d\bar{x}}{dt} = \frac{\Sigma\left(m\dfrac{dx}{dt}\right)}{\Sigma (m)}.$$

Now if the initial velocity of m_1 were resolvable into u_1, v_1, w_1, parallel to the axes respectively, and similarly for m_2, &c.

$$a = \frac{\Sigma (mu)}{\Sigma (m)}, \text{ and so for } b, \text{ \&c.}$$

306. II. CONSERVATION OF MOMENT OF MOMENTUM. Again, it is evident that if we multiply in succession equation (1) by y_1, and equation (2) by x_1, and subtract, and take the sum of all such remainders through the system of equations of the forms (1) and (2), we shall have

$$\Sigma\left[m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)\right]=0.$$

Or integrating once,

$$\Sigma\left[m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)\right]=A_3.$$

Now if in the plane of xy we take ρ, θ, the polar co-ordinates of the projection of m,

$$x\frac{dy}{dt}-y\frac{dx}{dt}=\rho^2\frac{d\theta}{dt};$$

$$\text{or }\ \Sigma\left(m\rho^2\frac{d\theta}{dt}\right)=A_3.$$

Now if a_z be the area swept out by the radius vector ρ on the plane of xy,

$$\tfrac{1}{2}\rho^2\frac{d\theta}{dt}=\frac{d(a_z)}{dt},$$

and our equation integrated gives

$$2\Sigma(ma_z)=A_3t,$$

no constant being necessary if we agree to reckon a_z from the position of ρ at time $t=0$.

This equation shews (since xy is *any* plane) that generally in the motion of a free system of particles, subject only to their mutual attractions, *the sum of the products of the mass of each particle of the system, into the area swept out by the radius vector of its projection on any plane, and about any point in that plane, will be proportional to the time.* See § 67.

Take a_x, a_y to represent for the planes yz, xz the same that a_z represent for xy,

$$2\Sigma(ma_x) = A_1 t,$$

$$2\Sigma(ma_y) = A_2 t.$$

The value of this quantity for a plane, the direction cosines of whose normal are λ, μ, ν, will be

$$(\lambda A_1 + \mu A_2 + \nu A_3)\, t,$$

and will be a maximum if

$$\lambda A_1 + \mu A_2 + \nu A_3 \text{ is so,}$$

subject to the equation of condition

$$\lambda^2 + \mu^2 + \nu^2 = 1.$$

This gives $\lambda = \dfrac{A_1}{\sqrt{(A_1^2 + A_2^2 + A_3^2)}} = \dfrac{A_1}{A}$ suppose,

with similar values for μ and ν;

and the value of the product for the plane so found is evidently

$$At.$$

Hence, we see also, that *the plane for which the sum of the products of the masses of the particles into the sectorial areas described by the radii vectores of their projections is a maximum, is a fixed plane or parallel to a fixed plane during the motion.* It has been called on this account the *Invariable Plane.*

307. III. Conservation of Energy. Multiply

$$(1) \text{ by } \frac{dx_1}{dt}, \quad (2) \text{ by } \frac{dy_1}{dt}, \quad (3) \text{ by } \frac{dz_1}{dt};$$

and, treating similarly all the other equations, add them all together.

Let us consider the result as regards the term on the right-hand side involving the product $m_p m_q$.

Written at length it is

$$\frac{m_p m_q \phi'({}_pr_q)}{{}_pr_q} \left\{(x_q - x_p)\frac{dx_p}{dt} + (x_p - x_q)\frac{dx_q}{dt}\right.$$

$$\left. + \text{ similar terms in } y \text{ and } z\right\};$$

and the portion in brackets is equal to

$$-\left\{(x_q - x_p)\frac{d}{dt}(x_q - x_p) + \text{similar terms in } y, z\right\};$$

$$\text{or, } -{}_pr_q \frac{d}{dt}({}_pr_q);$$

hence
$$\Sigma\left\{m\left(\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2}\right)\right\}$$

$$= -\Sigma\left\{m_p m_q \phi'({}_pr_q)\frac{d}{dt}({}_pr_q)\right\};$$

or, integrating,

$$\frac{1}{2}\Sigma(mv^2) = C - \Sigma\{m_p m_q \phi({}_pr_q)\}.$$

And by taking this integral between limits, we see that—*the change in the Vis Viva of the system in any time depends only on the relative distances of the particles at the beginning and end of that time*, § 73.

308. So far for the case of several particles. The simplest examples will of course be found in the case of two particles only, and to such we will confine our attention; as, when three or more are involved, the problem does not admit of exact solution, and in the two important applications which have been made of it, namely to the Lunar and Planetary Theories, it is found that a distinct method of approximation is required for each.

Since the acceleration of the center of inertia is zero, it follows that the motion of each particle with reference to that point is the same as if the latter were at rest. Also, if we apply to each particle of the system an acceleration equal

and opposite to that of any one of them, the latter will be reduced to rest, and the relative motion of the others about it will be unchanged. Hence, if there are only two, we see that the relative motion of one about the other will be the same as if the sum of the masses were substituted for the latter.

309. *Two particles, moving initially with given velocities in the same straight line, are subject to no forces but their mutual attraction which is inversely as the square of the distance; to determine the motion.*

The motion will evidently be confined to the straight line. Let m, m' be the masses of the particles estimated on the hypothesis that unit of mass exerts unit of force at unit of distance; x, x' their distances at any time t from a fixed point in the line of motion, then

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= \frac{mm'}{(x'-x)^2} \\ m'\frac{d^2x'}{dt^2} &= -\frac{mm'}{(x'-x)^2} \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (1).$$

Hence, if $\bar{x}$ be the co-ordinate of the center of gravity at time t,

$$m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} = (m+m')\frac{d^2\bar{x}}{dt^2} = 0,$$

$$m\frac{dx}{dt} + m'\frac{dx'}{dt} = (m+m')\frac{d\bar{x}}{dt} = C$$

$$= mV + m'V',$$

if V and V' be the initial velocities.

Integrating again,

$$mx + m'x' = (m+m')\bar{x} = (mV+m'V')\,t + C'$$

$$= (mV+m'V')\,t + ma + m'a' \dots\dots\dots (2),$$

if a, a' denote the initial positions of the particles.

Again, from equations (1),

$$\frac{d^2(x'-x)}{dt^2} = -\frac{m+m'}{(x'-x)^2},$$

from which, by multiplying by m or m', we see that the relative motion of the one with respect to the other, is the same as if the former had moved to the sum of the masses collected at the latter, and fixed.

Integrating once, we have

$$\left\{\frac{d(x'-x)}{dt}\right\}^2 = C + 2\frac{m+m'}{x'-x}.$$

At $t=0$, this is

$$(V'-V)^2 = C + 2\frac{m+m'}{a'-a};$$

and, eliminating C,

$$\left(\frac{d(x'-x)}{dt}\right)^2 = (V'-V)^2 + 2(m+m')\left\{\frac{1}{x'-x} - \frac{1}{a'-a}\right\}\ldots(3).$$

This is of the form

$$\left(\frac{d\omega}{dt}\right)^2 = \frac{A}{\omega} \pm B;$$

$$\text{or,}\ t = \int\frac{\sqrt{\omega}\,d\omega}{\sqrt{(A \pm B\omega)}},$$

which may be integrated by putting $\omega = y^2$. The integral will be circular or logarithmic according as B is $-$ or $+$. Thus we have $x'-x$ in terms of t, and knowing $mx+m'x'$ by (2), the motion is completely determined.

If at the instant of projection

$$(V-V')^2 = \frac{2(m+m')}{a'-a},$$

the formula (3) becomes

$$\sqrt{(x'-x)}\,\frac{d(x'-x)}{dt} = \pm\sqrt{\{2(m+m')\}},$$

$$\frac{2}{3}(x'-x)^{\frac{3}{2}} = C \pm \sqrt{\{2(m+m')\}}\, t,$$

$$\frac{2}{3}(a'-a)^{\frac{3}{2}} = C,$$

and the motion is completely determined.

310. There is another method of treating this problem. Suppose instead of determining the relative motion of the particles, we consider that of each relatively to the common center of inertia. The distance of m from the center of inertia is

$$\bar{x} - x = \frac{mx + m'x'}{m+m'} - x = \frac{m'(x'-x)}{m+m'};$$

and we easily find from (1),

$$m'\left(\frac{d^2x'}{dt^2} - \frac{d^2x}{dt^2}\right) = -\frac{mm'}{(x'-x)^2} - \frac{m'^2}{(x'-x)^2}.$$

Hence, for the relative motion of m and the center of inertia,

$$m\frac{d^2(\bar{x}-x)}{dt^2} = -\frac{mm'}{(x'-x)^2}$$

$$= -\frac{mm'^3}{(m+m')^2(\bar{x}-x)^2};$$

whence $\bar{x}-x$ may be determined, in finite circular or logarithmic terms, as before.

311. *Two particles, anyhow projected, are acted on solely by their mutual attraction; to shew that the line joining them is always parallel to a fixed plane.*

If m and m' be the particles, x, y, z, x', y', z', their co-ordinates at time t, r their distance, and P the mutual attraction, we have the following equations,

$$m\frac{d^2x}{dt^2} = P\frac{x'-x}{r}, \quad m'\frac{d^2x'}{dt^2} = P\frac{x-x'}{r},$$

with similar expressions for the other co-ordinates; hence

$$\frac{\frac{d^2(x'-x)}{dt^2}}{x'-x} = \frac{\frac{d^2(y'-y)}{dt^2}}{y'-y} = \frac{\frac{d^2(z'-z)}{dt^2}}{z'-z},$$

and integrating,

$$(x'-x)\frac{d(y'-y)}{dt} - (y'-y)\frac{d(x'-x)}{dt} = C_3,$$

with other two similar equations. Therefore

$$C_3(z'-z) + C_2(y'-y) + C_1(x'-x) = 0.$$

Hence, the line joining the particles is always parallel to the plane whose direction-cosines are as C_1, C_2, C_3. This corresponds to § 306.

Also it is evident that the motion of the particles with respect to each other in a plane parallel to this is the same as if the plane were at rest (§ 308).

From the preceding propositions the following are evident deductions.

The center of inertia of the two particles is at rest only when the initial velocities are zero, or when the directions of projection are the same or parallel, and the momenta equal and opposite.

The plane of relative motion will be at rest only when the initial directions lie in one plane.

If the force be inversely as the square of the distance, the relative orbits of the particles about each other, and therefore (§ 25) about their center of inertia, will be conic sections about a focus.

It is needless to pursue this any further, as the preceding results enable us to reduce the problem to cases treated of in former chapters.

312. *Two particles in space move under the action of given forces, as well as their mutual attraction; to determine the motion.*

Taking the same notation as in § 311, if X, Y, Z, X', Y', Z', be the resolved parts of the given forces on unit of mass, we have

$$m\frac{d^2x}{dt^2}=P\frac{x'-x}{r}+mX,\quad m'\frac{d^2x'}{dt^2}=P\frac{x-x'}{r}+m'X',$$

with similar equations for the other co-ordinates.

Hence, $$\frac{d^2(x'-x)}{dt^2}=-\frac{m+m'}{mm'}P\frac{x'-x}{r}+X'-X,$$

and so on. Thus we see that the relative motion about m will be found by applying to both particles (reversed in direction) the forces to whose action m is subjected.

Also, if $\bar{x}$, $\bar{y}$, $\bar{z}$, be the co-ordinates of the center of inertia, we find, from the above, three equations of the form

$$(m+m')\frac{d^2\bar{x}}{dt^2}=m\frac{d^2x}{dt^2}+m'\frac{d^2x'}{dt^2}=mX+m'X',$$

which shew that *the motion of the center of inertia is the same as if the particles had been collected into one, and acted on by the whole of the impressed forces.*

II. *Constrained Motion.*

313. Of the constrained motion of particles, we can only take particular examples, but there are some general considerations which deserve attention.

If two particles be connected by an inextensible string, its only effect is to prevent their relative distance becoming greater than its own length. If we introduce an unknown force T, for the tension of the string, the equations of motion can be written down, and the condition that the distance of the particles is equal to a given quantity will give us an additional equation, enabling us to eliminate, or to find the value of, this unknown force. If at any time the value of T becomes equal to zero, the motion of the particles must be investigated as if they were free, until the values of their co-ordinates shew that the string will begin to be tended again. In such a case, if their velocities resolved along the line joining them be not equal, an impact will take place, whose effects must be investigated by the methods of Chap. X.

When the particles are connected by a rigid rod without mass, we have an unknown reaction in the direction of the rod; and, to determine it, we have the geometrical condition that the distance between the particles is constant.

If there be more than two particles attached to the rod, it may exert a transverse force; but cases of this kind more properly belong to the Dynamics of a Rigid Body; and we therefore omit all consideration of them.

314. *Two particles, attached to each other by an inextensible string, are projected with given velocities in space; to determine the motion.*

We may without loss of generality consider the distance between the particles at the instant of projection, to be equal to the length of the string. If their velocities are wholly perpendicular to its direction, or if their resolved parts along it are equal and in the same direction, there will be no impact. If not, suppose the masses m and m' to have velocities v and v' parallel to the string at the instant it is stretched. It is evident that the impact will change each of these into $\dfrac{mv + m'v'}{m + m'}$. This then is determinate; so we may now in addition suppose the resolved parts of the velocities along the string equal to each other. Let x, y, z, x', y', z', be at any time the co-ordinates of the particles, then, if a be the length of the string,

$$m\frac{d^2x}{dt^2} = T\frac{x' - x}{a}, \quad m'\frac{d^2x'}{dt^2} = T\frac{x - x'}{a};$$

and so on.

Also, $$(x' - x)^2 + (y' - y)^2 + (z' - z)^2 = a^2,$$

which are seven equations to find T, and the six co-ordinates of m and m'. From the form of the equations, or by treating them as in § 311, we see that the string remains parallel to a fixed plane, and that the center of inertia moves uniformly in a straight line; the motion of the particles about each other, and about it, being the same as if it were at rest. Hence, the particles revolve with uniform angular velocity,

and the tension of the string is constant. From the above equations

$$T = \frac{mm'}{m+m'} \frac{V^2}{a},$$

where $V = \sqrt{\left[\left\{\frac{d(x'-x)}{dt}\right\}^2 + \left\{\frac{d(y'-y)}{dt}\right\}^2 + \left\{\frac{d(z'-z)}{dt}\right\}^2\right]}$,

is the relative velocity. The same result might have been easily obtained by considering that the velocity of m relative to the center of inertia is $\frac{m'V}{m+m'}$, that the radius of the circle it describes about that point is $\frac{m'a}{m+m'}$, and that T is the force which maintains it in that circle, and applying the last formula in § 135.

315. *Two particles, connected by an inextensible string which passes over a small smooth pulley, move under the action of gravity; to determine the motion.*

This was partly anticipated in § 285. Let m, m' be the masses, and let x, x' denote their distances from the pulley at time t. Then if T be the tension of the string (the same throughout since the pulley is smooth), we have

$$m\frac{d^2x}{dt^2} = mg - T,$$

$$m'\frac{d^2x'}{dt^2} = m'g - T.$$

But $x + x' =$ length of string $= a$ suppose. Hence supposing $m > m'$,

$$(m+m')\frac{d^2x}{dt^2} = (m-m')g \text{ .... } (1).$$

This equation completely determines the motion. Also, if we eliminate x and x', we have

$$T = \frac{2mm'}{m+m'} g,$$

and it is therefore constant.

This is one of the cases in which theoretical results may be tested by actual experiment with considerable accuracy. And it was this combination, with many delicate precautions against friction, &c. which Atwood made use of for experimental verification of the laws of motion.

We see, for instance, by equation (1), that we may easily keep $m+m'$ constant while $m-m'$ has any value, and thus by measuring the accelerations produced, find whether they are, in the same mass, proportional to the forces producing the motion. Again, keeping $m-m'$ constant, $m+m'$ may be varied at will. Hence by this process the second law of motion may be tested. See § 63. Again if, while the masses are in motion, a portion be suddenly removed from the greater so that they remain equal, (1) shews us that observation will enable us to test the first law of motion.

316. Instead of two masses connected by a string, suppose a flexible and uniform chain of length $2a$ hang over the pulley; then if x be the length hanging down on one side at time t, there will be $2a-x$ on the other, and the difference or

$$2(x-a),$$

is the portion whose weight accelerates the motion. Hence, μ being the mass of the chain per unit of length, we have

$$2\mu a \frac{d^2x}{dt^2} = 2\mu g(x-a);$$

which gives $$x - a = A\epsilon^{\sqrt{\frac{g}{a}}t} + B\epsilon^{-\sqrt{\frac{g}{a}}t}.$$

If the chain were initially at rest, a portion $a+b$ being on one side of the pulley,

$$b = A + B,$$
$$0 = A - B;$$

$$\therefore\ x - a = \frac{b}{2}\left(\epsilon^{\sqrt{\frac{g}{a}}t} + \epsilon^{-\sqrt{\frac{g}{a}}t}\right).$$

This is true until $x = 2a$, that is, till the chain leaves the pulley; the value of t at that instant being t_0, we have

$$\frac{2a}{b} = \epsilon^{\sqrt{\frac{g}{a}}t_0} + \epsilon^{-\sqrt{\frac{g}{a}}t_0};$$

and therefore $t_0 = \sqrt{\frac{a}{g}}\log\left\{\frac{a}{b} + \sqrt{\left(\frac{a^2}{b^2} - 1\right)}\right\}$.

If, for example, $b = \frac{3a}{5}$, i.e. if the portions of the chain were initially as 4 : 1,

$$t_0 = \sqrt{\frac{a}{g}}\log_\epsilon 3.$$

317. *A particle, of mass* m, *hangs over a small pulley, and the other end of the string is attached to a mass* m′ *lying on a smooth horizontal plane; to determine the motion.*

Let a be the length of the string, then it is evident that if b be the height of the pulley above the plane, θ the angle the string attached to m' makes with the vertical, and x the distance of m from the pulley, we must have

$$x + \frac{b}{\cos\theta} = a \text{ .... (1).}$$

Also, if T be the tension of the string, the equations of motion are

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= mg - T \\ m'\frac{d^2(b\tan\theta)}{dt^2} &= -T\sin\theta \end{aligned}\right\} \text{ .... (2).}$$

These will suffice for the determination of the motion, and the tension of the string; unless $T\cos\theta$ should become greater than $m'g$, in which case the mass m' will be withdrawn from

the plane, and instead of the last of equations (2) we must have two equations for the determination of its motion.

318. *Two particles* m, m′, *connected by a rigid rod, are forced to move one on each of two straight lines in a vertical plane, inclined at angles* α, α' *to the vertical; to determine the motion, and the time of a small oscillation about the position of equilibrium.*

Let θ be the inclination of the rod to the vertical at time t, T its tension, R, R' the reactions of the lines, x, x' the distances of the particles from the point of intersection of the lines, and a the length of the rod.

Then,

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= mg\cos\alpha + T\cos(\theta+\alpha) \\ m'\frac{d^2x'}{dt^2} &= m'g\cos\alpha' - T\cos(\theta-\alpha') \end{aligned}\right\} \ldots\ldots\ldots\ldots (1),$$

$$\left.\begin{aligned} R &= mg\sin\alpha + T\sin(\theta+\alpha) \\ R' &= m'g\sin\alpha' + T\sin(\theta-\alpha') \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (2),$$

$$\left.\begin{aligned} x &= x'\cos(\alpha+\alpha') - a\cos(\theta+\alpha) \\ x' &= x\cos(\alpha+\alpha') + a\cos(\theta-\alpha') \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (3).$$

These six equations give x, x', R, R', T and θ in terms of t, and thus theoretically complete the solution.

319. For the time of a small oscillation, we must first find the position of equilibrium. This will, of course, be obtained by equating to zero the right-hand members of equations (1). (§ 64). Let $\bar{x}$, $\bar{x}'$, correspond to this position; and let

$$x = \bar{x} + \xi, \quad x' = \bar{x}' + \xi',$$

where ξ and ξ' are infinitely small.

Eliminating T from equations (1), and then θ by means of (3), we have, putting $\beta = \alpha + \alpha'$,

$$m\left(\frac{d^2x}{dt^2}-g\cos\alpha\right)(x'-x\cos\beta)-m'\left(\frac{d^2x'}{dt^2}-g\cos\alpha'\right)(x-x'\cos\beta)=0.$$

But $x^2+x'^2-2xx'\cos\beta=a^2$;

from which, neglecting quantities of the second order,

$$\bar{x}\xi+\bar{x}'\xi'-(\bar{x}\xi'+\bar{x}'\xi)\cos\beta=0;$$

$$\text{or,}\quad \xi'=\xi\,\frac{\bar{x}-\bar{x}'\cos\beta}{\bar{x}\cos\beta-\bar{x}'}.$$

Hence, eliminating x, x', and ξ' from these equations, we have

$$m\left(\frac{d^2\xi}{dt^2}-g\cos\alpha\right)\left\{\bar{x}'-\bar{x}\cos\beta+\xi\left(\frac{\bar{x}-\bar{x}'\cos\beta}{\bar{x}\cos\beta-\bar{x}'}-\cos\beta\right)\right\}$$

$$-m'\left(\frac{d^2\xi}{dt^2}\,\frac{\bar{x}-\bar{x}'\cos\beta}{\bar{x}\cos\beta-\bar{x}'}-g\cos\alpha'\right)\times$$

$$\left\{\bar{x}-\bar{x}'\cos\beta+\xi\left(1-\frac{\bar{x}-\bar{x}'\cos\beta}{\bar{x}\cos\beta-\bar{x}'}\cos\beta\right)\right\}=0.$$

Keeping only terms of the first order in ξ, since the terms not involving ξ or its differential coefficient must evidently vanish of themselves; we have

$$\{m(\bar{x}'-\bar{x}\cos\beta)^2+m'(\bar{x}-\bar{x}'\cos\beta)^2\}\frac{d^2\xi}{dt^2}$$

$$+g\sin^2\beta\,(m\bar{x}\cos\alpha+m'\bar{x}'\cos\alpha')\,\xi=0,$$

an equation of the form

$$\frac{d^2\xi}{dt^2}+n^2\xi=0;$$

and the time of oscillation is $\frac{2\pi}{n}$. § 125.

320. *Two particles,* m *and* m', *are attached at different points to an inextensible string, one of whose extremities is fixed. If the system be displaced, to determine the motion.*

Take the axes of x and y horizontal, and that of z vertically downwards, the extremity of the string being origin.

Let a, a' be the lengths of the portions of the string, θ, θ' the angles they make with the vertical, ϕ, ϕ' the angles which vertical planes through them at time t, make with the plane of xz. Let x, y, z, x', y', z', be the co-ordinates of the particles and T, T' the tensions of the strings.

Then

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= -T\sin\theta\cos\phi + T'\sin\theta'\cos\phi', \\ m\frac{d^2y}{dt^2} &= -T\sin\theta\sin\phi + T'\sin\theta'\sin\phi', \\ m\frac{d^2z}{dt^2} &= mg - T\cos\theta + T'\cos\theta', \end{aligned}\right\}$$

$$\left.\begin{aligned} m'\frac{d^2x'}{dt^2} &= -T'\sin\theta'\cos\phi', \\ m'\frac{d^2y'}{dt^2} &= -T'\sin\theta'\sin\phi', \\ m'\frac{d^2z'}{dt^2} &= m'g - T'\cos\theta'. \end{aligned}\right\}$$

Besides these, we have the six equations for x, y, z, x', y', z' in terms of a, a', θ, ϕ, θ', ϕ', in all twelve equations for the determination of the twelve unknowns in terms of t.

321. These equations will be much simplified if we consider the displacement to be in one plane, as the motion will evidently be confined to that plane. By this means we at once get rid of y, y', ϕ and ϕ'. A still greater simplification will be obtained by taking in addition the condition that θ and θ' are so small, that their squares and higher powers may be neglected. With these our equations become

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= -T\theta + T'\theta', \\ m\frac{d^2z}{dt^2} &= mg - T + T' \end{aligned}\right\}$$

$$\left. \begin{aligned} m' \frac{d^2 x'}{dt^2} &= - T' \theta', \\ m' \frac{d^2 z'}{dt^2} &= m' g - T'. \end{aligned} \right\}$$

And
$$x = a\theta,$$
$$x' = a\theta + a'\theta',$$
$$z = a,$$
$$z' = a + a'.$$

Hence, $T' = m'g$, and $T = (m + m')\, g$,

$$\left. \begin{aligned} ma \frac{d^2\theta}{dt^2} &= - (m + m')\, g\theta + m' g\theta', \\ m' \left(a \frac{d^2\theta}{dt^2} + a' \frac{d^2\theta'}{dt^2} \right) &= - m' g\theta' \end{aligned} \right\}$$

Introducing an indeterminate multiplier, and adding,

$$(m + \lambda m') \frac{d^2\theta}{dt^2} + \lambda m' \frac{a'}{a} \frac{d^2\theta'}{dt^2} + \frac{g}{a} \{(m + m')\, \theta + m' (\lambda - 1)\, \theta'\} = 0.$$

Let λ_1, λ_2 be the roots of the equation

$$\frac{\lambda}{m + \lambda m'} \frac{a'}{a} = \frac{\lambda - 1}{m + m'}.$$

Evidently one is positive and the other negative, and the form of the equation shews that for both $m + \lambda m'$ is positive.

Put $\phi = \theta + \dfrac{\lambda m'}{m + \lambda m'} \dfrac{a'}{a} \theta' = \theta + k\theta'$, suppose.

Then the above equation gives

$$\frac{d^2\phi}{dt^2} + \frac{g}{a} \frac{m + m'}{m + \lambda m'} \phi = 0.$$

By the recent remark the coefficient of ϕ is positive for

both values of λ; let its values be n_1^2 and n_2^2, and we have, k_1, ϕ_1, k_2, ϕ_2, being the corresponding values of k and ϕ,

$$\phi_1 = \theta + k_1 \theta' = \alpha_1 \cos(n_1 t + \beta_1),$$

$$\phi_2 = \theta + k_2 \theta' = \alpha_2 \cos(n_2 t + \beta_2),$$

where α_1, α_2, β_1, β_2, are arbitrary constants.

Hence,

$$\theta = \frac{1}{k_2 - k_1} \{k_2 \alpha_1 \cos(n_1 t + \beta_1) - k_1 \alpha_2 \cos(n_2 t + \beta_2)\},$$

$$\theta' = \frac{1}{k_1 - k_2} \{\alpha_1 \cos(n_1 t + \beta_1) - \alpha_2 \cos(n_2 t + \beta_2)\}.$$

Having given the initial values of θ, θ', $\frac{d\theta}{dt}$ and $\frac{d\theta'}{dt}$, we find α_1, α_2, β_1, β_2, and thus the solution is complete. It may be noticed that the values of θ and θ' may be found at any time by taking the algebraic sum of the corresponding values of the inclinations to the vertical of two pendulums whose times of oscillation are $\frac{2\pi}{n_1}$ and $\frac{2\pi}{n_2}$. Also, if n_1, n_2 be commensurable, the system will in time return to its first position, and the motion will be periodic.

A very slight modification of the process gives us the result of *small* displacements not in one plane: but the student may easily work out these for himself.

We have here a simple example of the principle of the "Coexistence of Small Oscillations;" but this, for its satisfactory treatment in the general case, requires the use of D'Alembert's Principle; which, though (§ 69) merely a corollary to the Third Law of Motion, and as such clearly pointed out by Newton, is beyond the professed limits of the present treatise.

322. The examples, which have just been given, may suffice to convey an idea of the mode of applying our methods to any proposed case of motion of two constrained particles.

These methods are applicable to more complicated cases, when more particles than two are involved; but nothing would be gained by such a proceeding, as D'Alembert's Principle supplies us with a far simpler mode of investigating the motions of any system of free or connected particles: especially when it is simplified in its application by the beautiful system of *Generalized Co-ordinates* introduced by Lagrange. See Thomson and Tait's *Natural Philosophy*, § 329.

EXAMPLES.

(1) Two spheres whose masses are M and M' are placed in contact, and one of them is projected in the line of centers with velocity V. If the law of attraction be D^{-2}, find where, and after what time, they will meet.

(2) If the sun were broken up into an indefinite number of fragments, uniformly filling the sphere of which the earth's orbit is a great circle, shew that each would revolve in a year.

(3) A thin spherical shell of mass M is driven out symmetrically by an internal explosion. Shew that if, when the shell has a radius a, the outward velocity of each particle be v, the fragments can never be collected by their mutual attraction unless

$$v^2 < \frac{M}{a}.$$

(4) Two equal particles are initially at rest in two smooth tubes at right angles to each other. Shew that whatever were their positions, and whatever their law of attraction, they will reach the intersection of the tubes together.

(5) In last question suppose the original distances from the intersection of the tubes to be a, b, and the attraction as the square of the distance inversely, find the future paths if at any instant the constraint is removed.

(6) A number of equal particles, attracting each other

directly as the distance, are constrained to move in parallel tubes; if the positions of the particles be given at the commencement of the motion, determine the subsequent motion of each; and shew that the particles will oscillate symmetrically with respect to the plane perpendicular to the tubes which passed through their center of inertia at the commencement of the motion.

(7) Two given masses are connected by a slightly elastic string, and projected so as to whirl round, find the time of a small oscillation in the length of the string.

Give a numerical result, supposing the masses to weigh 1 lb. and 2 lbs. respectively, and the natural length of the string to be 1 yard, and supposing that it stretches $\frac{1}{10}$th inch for a tension of 1 lb.

(8) Two equal masses M, are connected by a string which passes through a hole in a smooth horizontal plane. One of them hanging vertically, shew that the other describes on the plane a curve whose differential equation is

$$\frac{d^2u}{d\theta^2} + \frac{u}{2} - \frac{g}{2h^2u^2} = 0,$$

and that the tension of the string is

$$M\frac{g + h^2u^3}{2}.$$

(9) Two equal particles connected by a string are placed in a circular tube. In the circumference is a center of force $\propto \frac{1}{D}$. One particle is initially at its greatest distance from the center of force, shew that if v, v' be the velocities with which they pass through a point 90° from the center of force,

$$\epsilon^{-\frac{v^2}{\mu}} + \epsilon^{-\frac{v'^2}{\mu}} = 1.$$

(10) Two equal balls repelling each other with a force $\propto \frac{1}{D^2}$ hang from the same point by strings of length l. Shew

that if when in equilibrium, the strings making an angle 2α with each other, they be approximated by equal small arcs, the time of an oscillation is the same as that of a pendulum whose length is

$$\frac{l\cos\alpha}{1+2\cos^2\alpha}.$$

(11) One of two equal particles connected by an inelastic string moves in a straight groove. The other is projected parallel to the groove, the string being stretched; determine the motion, and shew that the greatest tension is four times the least.

(12) Two particles are connected by an elastic string of length $2a$, and one is projected perpendicularly to the string when it is unstretched. Shew that in the relative orbit

$$\frac{1}{a^2}-\frac{1}{p^2}\propto (r-a)^2.$$

(13) Two equal particles connected by a rigid rod move on a vertical circle. If they be slightly disturbed from the higher position of equilibrium, determine the motion.

Also find the time of a small oscillation about the position of stable equilibrium.

(14) Two particles P and Q are connected by a rigid rod. P is constrained to move in a smooth horizontal groove. If the particles be initially at rest, PQ making a given angle with the groove in a vertical plane through it, find the velocity of Q when it reaches the groove, and shew that Q's path in the vertical plane is an ellipse.

(15) A particle of mass m has attached to it two equal weights m' by means of strings passing over pulleys in the same horizontal plane, and is initially at rest halfway between them. Determine the motion. Shew that if the distance between the pulleys be $2a$, the velocity of m will be zero when it has fallen through a space

$$\frac{4mm'a}{4m'^2-m^2},$$

and that it will have its greatest velocity when passing through its position of statical equilibrium. (§ 73.)

(16) Two masses M, M' are connected by a string which passes over a smooth peg. To M' is attached a string which supports a mass m such that $M'+m=M$, and m is displaced through an angle α. Investigate the motion, supposing m so small that the horizontal motion of M' may be neglected. Shew that the string $M'm$ will be vertical after the time

$$\left(\frac{\lambda}{g}\right)^{\frac{1}{2}}\int_0^\alpha\left(\frac{1-\frac{m}{2M}\sin^2\theta}{\cos\theta-\cos\alpha}\right)^{\frac{1}{2}}d\theta,$$

where λ is the length of $M'm$.

GENERAL EXAMPLES.

(1) A spiral spring is stretched an inch by each additional pound appended to its lower end; find the greatest velocity which will be acquired by a mass of 20 lbs. appended to the unstretched spring and allowed to fall.

Also find how far the mass will fall, and the time of a complete oscillation.

(2) Find the form of the hodograph, and the law of its description, for any point of one circular disc rolling uniformly on another. Hence, find the force under which a free particle will describe an epitrochoid, as it is described by a point of the uniformly rolling disc.

(3) Form the equation to the surfaces of equal *time*, as those of equal *action* were found in § 254.

(4) Apply a method similar to that of § 255 to find the equation to the common brachistochrone.

(5) Find the law of the force when the brachistochrone is an ellipse with the center of force in its focus.

(6) A rod slides between two rough parallel horizontal bars, in a plane perpendicular to the bars: determine the

motion while it is rectilinear, but neither horizontal nor vertical.

(7) Determine the (unresisted) motion of a mass projected vertically at a given point of the earth's surface with a velocity of 7 miles per second.

(8) Apply the principle of varying action to the determination of the (unresisted) motion of a projectile.

(9) Shew that the *action* and *time*, in any arc of the ordinary brachistochrone commencing at the cusp, are represented by the *area* and *arc* of the corresponding segment of the generating circle.

(10) In the parabolic motion of a projectile, the *action* is represented by the area included between the curve, the directrix, and the two vertical ordinates: and the *time* by the intercept on the directrix.

(11) Given a central orbit, and the law of its description, find the differential equation of a curve such that if tangents be drawn to it from any two points of the orbit, the action shall be represented by the area included by these tangents and the two curves.

(12) A particle moves in a given line, under the action of a force $= -\mu s - f\frac{ds}{dt}$; and a given impulse acts on it, alternately in opposite directions in its line of motion, at intervals each equal to T. Find the resultant periodic motion. (This is the general case of the pendulum of an electrically-controlled clock.)

APPENDIX.

A. *On the integration of the equations of motion about a center of force.*

In general, (Chap. V.) the problem of central forces is solved by considering the equation connecting u $\left(\text{or } \frac{1}{r}\right)$ and θ, and employing the resulting integrated relation between r and θ to find θ in terms of t from the law of equable description of areas. If we try to express r and θ separately, in terms of t, without first determining the form of the orbit, we are led to a host of curious results which may be easily obtained; so easily indeed, that we shall merely notice one or two of them.

From the usual equations for motion about a center, *i.e.*,

$$\frac{d^2x}{dt^2} = -P\frac{x}{r},$$

$$\frac{d^2y}{dt^2} = -P\frac{y}{r},$$

where P is the acceleration due to the central force, we get at once

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = -2\int Pdr,$$

and $$x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} = -Pr.$$

Adding, we have immediately,

$$\frac{d}{dt}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{1}{2}\frac{d^2(r^2)}{dt^2} = -2\int Pdr - Pr \dots (1).$$

This, for any assigned form of P in terms of r, will evidently give us r^2 in terms of t.

Now there is a remarkable case in which r^2 can be generally expressed as a rational integral function of t. Suppose

$$-2\int Pdr - Pr = C \text{ .... } (2),$$

and we have

$$\frac{d^2(r^2)}{dt^2} = 2C,$$

or

$$r^2 = A + 2Bt + Ct^2 \text{ .... } (3).$$

From (2) we find by differentiation

$$3P + r\frac{dP}{dr} = 0,$$

or

$$P \propto \frac{1}{r^3}.$$

Hence the case in question is that of the inverse third power. It may be worth while to find θ in terms of t, and to obtain, by elimination of t, the equations to the orbits which are possible with such a force.

We have, in all central orbits,

$$r^2\frac{d\theta}{dt} = h \text{ .... } (4).$$

Hence, in the present case, by (3),

$$\frac{d\theta}{dt} = \frac{h}{A + 2Bt + Ct^2} = \frac{h}{C} \cdot \frac{1}{\left(t + \frac{B}{C}\right)^2 + \frac{AC - B^2}{C^2}} \text{...} (5).$$

Put now
$$\tau = t + \frac{B}{C},$$

and we get
$$r^2 = C\left\{\tau^2 + \frac{AC - B^2}{C^2}\right\} \text{ .... } (3'),$$

and $$\frac{d\theta}{d\tau} = \frac{h}{C} \frac{1}{\tau^2 + \frac{AC - B^2}{C^2}} \text{..............} (5').$$

There are, of course, four cases.

I. $AC = B^2$. The integral of (5′) is

$$\theta + \alpha = -\frac{h}{C}\frac{1}{\tau};$$

and $$r = \pm\sqrt{C}\tau.$$

Here C must be *positive*. Hence

$$r = \mp \frac{h}{\sqrt{C}(\theta + \alpha)},$$

the reciprocal spiral.

II. $\frac{AC - B^2}{C^2} = a^2$. (3′) and (5′) give

$$\frac{aC}{h}(\theta + \alpha) = \tan^{-1}\frac{\tau}{a},$$

and therefore $$r^2 = Ca^2 \sec^2 \frac{aC}{h}(\theta + \alpha),$$

or $$r \cos \frac{aC}{h}(\theta + \alpha) = \sqrt{\frac{AC - B^2}{C}}$$

III. $\frac{AC - B^2}{C^2} = -a^2$. Here

$$\frac{C}{h}(\theta + \alpha) = \frac{1}{2a}\log\frac{\tau - a}{\tau + a},$$

and $$r^2 = C(\tau^2 - a^2),$$

whence, after a reduction or two,

$$r = \frac{2a\sqrt{C}}{\epsilon^{\frac{aC}{h}(\theta+\alpha)} - \epsilon^{-\frac{aC}{h}(\theta+\alpha)}} = \frac{1}{M\epsilon^{\frac{aC}{h}\theta} - N\epsilon^{-\frac{aC}{h}\theta}}.$$

IV. $C=0$,

$$\theta = \frac{h}{2B}\log\frac{r}{M}, \text{ or } r = M\epsilon^{\frac{2B}{h}\theta}.$$

These are, of course, the results of the integration of the usual equation between u and θ. (Compare Chap. V. Ex. (15)).

As another case, suppose in (1),

$$-2\int P dr - Pr = mr^2 + \frac{C}{2} \qquad (6).$$

Differentiate, multiply by r^2, and integrate, then

$$P = -\tfrac{1}{2}mr + \frac{n}{r^3}.$$

Hence, in the case of the direct first power, or a combination of this with the inverse third,

$$\frac{d^2(r^2)}{dt^2} = 2mr^2 + C,$$

which gives, according as m is positive or negative,

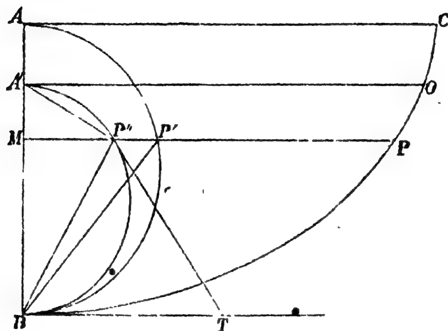
$$r^2 + \frac{C}{2m} = \begin{Bmatrix} M\epsilon^{\sqrt{(2m)}t} + N\epsilon^{-\sqrt{(2m)}t} \\ M\cos\{\sqrt{(-2m)}\,t + N\} \end{Bmatrix}.$$

By means of (4), these equations give us θ in terms of t, and, the latter being eliminated, we have the required orbit, which becomes the ellipse or hyperbola as usual when $n=0$, it being observed that we have an additional disposable constant introduced by the method employed in obtaining equation (1). It is evident that results of this kind may be multiplied indefinitely. To classify the cases in which the equations for r^2 and θ in terms of t can be completely integrated would be an interesting, but by no means an easy problem.

The method here employed is interesting as being that which is at once suggested by the application of Quaternions to the problem of Central Orbits.

B. *To find the time of fall from rest down any arc of an inverted cycloid.*

Let O be the point from which the particle commences its motion. Draw OA' parallel to CA, and on BA' describe



a semicircle. Let P, P', P'' be corresponding points of the curve, the generating circle, and the circle just drawn, and let us compare the velocities of the particle at P, and the point P''. Let $P''T$ be the tangent at P''.

$$\frac{\text{velocity of } P''}{\text{velocity of } P} = \frac{\text{element at } P''}{\text{element at } P}$$

$$= \frac{P''T}{BP'} = \frac{P''T}{BP''}\sqrt{\frac{A'B}{AB}}$$

$$= \frac{A'B}{2A'P''}\sqrt{\frac{A'B}{AB}}.$$

But velocity of $P = \surd(2g \,.\, A'M) = \sqrt{\frac{2g}{A'B}} \,.\, A'P''$.

Hence velocity of $P'' = \sqrt{\frac{g}{2AB}} \,.\, A'B$, a constant.

And, as the length of $A'P''B$ is $\frac{\pi}{2} \cdot A'B$,

time from A' to B in circle = time from O to B in cycloid

$$= \pi \sqrt{\frac{AB}{2g}}.$$

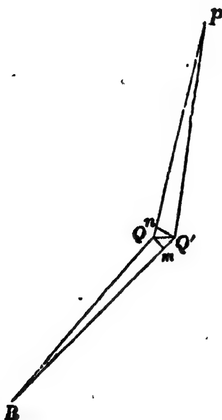
COR. It is evident from the proof, that the particle descends half the vertical space to B in half the time it takes to reach B.

C. *To find the nature of the brachistochrone, gravity being the only impressed force.*

The following is founded on Bernoulli's original solution. (WOODHOUSE, *Isoperimetrical Problems.*)

From Art. 182 it is evident that the curve lies in the vertical plane which contains the given points. Also it is easy to see that if the time of descent through the entire curve is a minimum, that through any portion of the curve is less than if that portion were changed into any other curve.

And it is obvious that, *between any two contiguous equal values of a continuously varying quantity, a maximum or*



minimum must lie. [This principle, though excessively simple (witness its application to the barometer or thermometer), is of very great power, and often enables us to solve problems of maxima and minima, such as require in analysis not merely the processes of the Differential Calculus, but those of the Calculus of Variations. The present is a good example.]

Let, then, PQ, QR and PQ', $Q'R$ be two pairs of indefinitely small sides of a polygon such that the time of descending through either pair, starting from P with a given velocity, may be *equal*. Let QQ' be horizontal and indefinitely small compared with PQ and QR. The brachistochrone must lie *between* these paths, and must possess any property which they possess in common. Hence if v be the velocity down PQ (supposed uniform) and v' that down QR, drawing Qm, $Q'n$ perpendicular to RQ', PQ, we must have

$$\frac{Qn}{v} = \frac{Q'm}{v'}.$$

Now if θ be the inclination of PQ to the horizon, θ' that of QR, $Qn = QQ' \cos\theta$, $Q'm = QQ' \cos\theta'$. Hence the above equation becomes

$$\frac{\cos\theta}{v} = \frac{\cos\theta'}{v'}.$$

This is true for any two consecutive elements of the required curve; therefore throughout the curve

$$v \propto \cos\theta.$$

But $v^2 \propto$ vertical space fallen through (§ 171). Hence the curve required is such that the cosine of the angle it makes with the horizontal line through the point of departure varies as the square root of the distance from that line; which is easily seen to be a property of the cycloid, if we remember that the tangent to that curve is parallel to the corresponding chord of its generating circle. For in the fig. p. 163,

$$\cos OP'N = \cos OAP' = \frac{AP'}{AO} = \sqrt{\frac{AN}{AO}} \propto \sqrt{AN}.$$

The brachistochrone then, in the case of gravity being the

only impressed force, is an inverted cycloid whose cusp is at the point from which the particle descends.

C_1. Were there any number of impressed forces we might suppose their resultant constant in magnitude and direction for two successive elements. Then reasoning similar to that in § 182 would shew that the osculating plane to the brachistochrone always contains the resultant force. Again we should have as in last Article,

$$\frac{\cos\theta}{v} = \frac{\cos\theta'}{v'},$$

where θ is now the complement of the angle between the curve and the resultant of the impressed forces.

Let that resultant $= F$, and let the element $PQ = \delta s$, and $\theta' = \theta + \delta\theta$. Then since F is supposed the same at P and Q,

$$v'^2 - v^2 = 2F\delta s \sin\theta \text{ (by Chap. IV.)},$$

$$\text{or } v\delta v = F\delta s \sin\theta.$$

But $v \propto \cos\theta$; which gives

$$\frac{\delta v}{v} = -\frac{\sin\theta}{\cos\theta}\delta\theta.$$

Hence $$\frac{v^2}{\frac{\delta s}{\delta\theta}} = -F\cos\theta.$$

But in the limit $\frac{\delta s}{\delta\theta} = \rho$, the radius of absolute curvature at Q, and $F\cos\theta$ is the normal component of the impressed force. Hence in the general brachistochrone the pressure due to centrifugal force equals that due to the impressed forces, our result of § 189.

C_2. Now for the unconstrained path from P to R we have $\int v ds$ a minimum. Hence in the same way as before, ϕ being the angle corresponding to θ, $v\cos\phi = v'\cos\phi'$ from element to element, and therefore throughout the curve, if the *direction* of the force be constant.

But in the brachistochrone,

$$\frac{\cos\theta}{v} = \frac{\cos\theta'}{v'}.$$

Now if the velocities in the two paths be equal at any equipotential surface, they will be equal at every other. Hence taking the angles for any equipotential surface

$$\cos\theta\cos\phi = \text{constant}.$$

As an example, suppose a parabola with its vertex upwards to have for directrix the base of an inverted cycloid; these curves evidently satisfy the above condition, the one being the free path, the other the brachistochrone, for gravity, and the velocities being in each due to the same horizontal line. And it is seen at once that the product of the cosines of the angles which they make with any horizontal straight line which cuts both is a constant whose magnitude depends on that of the cycloid and parabola, its value being $\sqrt{\frac{l}{4a}}$ where l is the latus rectum of the parabola, and a the diameter of the generating circle of the cycloid.

D. *To shew that of two curves both concave in the sense of gravity, joining the same points in a vertical plane and not meeting in any other point, a particle will descend the enveloped in less time than it will the enveloping curve; the initial velocity being the same in both cases.*

Take the axis of x as the line to a fall from which the initial velocity is due, and the axis of y in the sense of gravity, then

$$\frac{ds}{dt} = \sqrt{(2gy)};$$

$$\therefore t_1 = \int_{x_1}^{x_2} \frac{\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\,dx}{\sqrt{(2gy)}}$$

$$\propto \int_{x_1}^{x_2} \frac{\sqrt{(1+p^2)}}{\sqrt{y}}\,dx \quad\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

$$\therefore\ \delta t_1 \propto \int_{x_1}^{x_2} dx\,\delta y\left(N - \frac{dP}{dx}\right),$$

(since the limits are constant),

$$\propto \int_{x_1}^{x_2} \delta y\,dx\left[-\tfrac{1}{2}\frac{\sqrt{(1+p^2)}}{y\sqrt{y}} + \tfrac{1}{2}\frac{p^2}{y\sqrt{y}\sqrt{(1+p^2)}} - \frac{q}{\sqrt{y}\sqrt{(1+p^2)}} + \frac{p^2 q}{\sqrt{y}\,(1+p^2)^{\frac{3}{2}}}\right]$$

$$\propto \int_{x_1}^{x_2} \delta y\,dx\,\frac{-(1+p^2) + 2p^2qy - 2qy\,(1+p^2)}{y\sqrt{y}\,(1+p^2)^{\frac{3}{2}}}$$

$$\propto \int_{x_1}^{x_2} \delta y\,dx\,\frac{-(1+p^2+2yq)}{y\sqrt{y}\,(1+p^2)^{\frac{3}{2}}}.$$

Now, the curve is convex to the axis of x, hence yq is positive, and by (1) $\sqrt{y}$ and $\sqrt{(1+p^2)}$ have the same sign. Hence the sign of δt_1 is the opposite of that of δy, and for an enveloping curve δy is negative. Hence the time of fall will be longer.

We may thus pass from one curve to any other enveloping one, even situated at a finite distance, provided the latter be concave throughout; else the multiplier of $\delta y\,.\,dx$ in the integral might change sign between the limits. (BERTRAND, *Liouville's Journal*, Vol. VII.)

A simple geometrical proof of this theorem may easily be obtained by drawing successive normals to the inner curve and producing them to meet the outer. The velocities in the pairs of arcs, thus cut out of the two curves, are equal (if the curves be indefinitely close), but the arcs themselves are generally longer in the outer curve, since the convexity of the inner curve is everywhere turned to it.

E. *To find the curve in which the time of descent to the lowest point is a given function* ϕ (a) *of* a, *the vertical height fallen through.*

$$\text{Here } \sqrt{(2g)}\,t = \phi\,(a) = \int_0^a \frac{ds}{\sqrt{(a-x)}}.$$

Hence, the problem may be thus stated,

Given $$\phi(a) = \int_0^a \frac{ds}{\sqrt{a-x}},$$

where ϕ is a known function, find s *in terms of* x.

Consider the integral (GREGORY'S *Examples*, p. 471)

$$\int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\,dy = \frac{\Gamma\alpha\,\Gamma\beta}{\Gamma(\alpha+\beta)},$$

(where α and β are each positive).

Let $\beta = 1 - n$, where $n < 1$, then

$$\int_0^1 \frac{y^{\alpha-1}\,dy}{(1-y)^n} = \frac{\Gamma\alpha\,\Gamma(1-n)}{\Gamma(\alpha+1-n)}.$$

Next put $y = \frac{z}{a}$, the limits are 0 and a, and

$$\int_0^a \frac{\left(\frac{z}{a}\right)^{\alpha-1}\frac{dz}{a}}{\left(1-\frac{z}{a}\right)^n} = \int_0^a \frac{a^{n-\alpha}\,z^{\alpha-1}\,dz}{(a-z)^n} = \frac{\Gamma\alpha\,\Gamma(1-n)}{\Gamma(\alpha+1-n)};$$

or $$\int_0^a \frac{z^{\alpha-1}\,dz}{(a-z)^n} = \frac{\Gamma\alpha\,\Gamma(1-n)}{\Gamma(\alpha+1-n)}\,a^{\alpha-n}.$$

Hence

$$\frac{da}{(x-a)^{1-n}}\int_0^a \frac{z^{\alpha-1}\,dz}{(a-z)^n} = \frac{\Gamma\alpha\,\Gamma(1-n)}{\Gamma(\alpha+1-n)}\int_0^x \frac{a^{\alpha-n}\,da}{(x-a)^{1-n}}.$$

But, putting $a = \xi x$, we have

$$\int_0^x \frac{a^{\alpha-n}\,da}{(x-a)^{1-n}} = \int_0^1 \frac{(\xi x)^{\alpha-n}\,x\,d\xi}{(x-\xi x)^{1-n}} = x^\alpha \int_0^1 \frac{\xi^{\alpha-n}\,d\xi}{(1-\xi)^{1-n}}$$

$$= x^\alpha\,\frac{\Gamma(\alpha-n+1)\,\Gamma n}{\Gamma(\alpha+1)};$$

by the formula already cited.

Hence

$$\int_0^x \frac{da}{(x-a)^{1-n}} \int_0^a \frac{z^{a-1}\,dz}{(a-z)^n} = x^a \frac{\Gamma a\,\Gamma(1-n)\,\Gamma(n)}{\Gamma(a+1)}$$

$$= x^a \frac{1}{a}\frac{\pi}{\sin n\pi}.$$

Now, let

$$\int \phi(\alpha)\,x^\alpha\,d\alpha = f(x),$$

whence
$$\int \phi(\alpha)\,\alpha x^{\alpha-1}\,d\alpha = f'(x),$$

and
$$\int \phi(\alpha)\,\alpha z^{\alpha-1}\,d\alpha = f'(z);$$

then multiply the terms of the last equation by $\alpha\phi(\alpha)\,d\alpha$, and integrate. We have

$$\int_0^x \frac{da}{(x-a)^{1-n}} \int_0^a \frac{\int \alpha\phi(\alpha)\,z^{\alpha-1}\,d\alpha\,.\,dz}{(a-z)^n} = \frac{\pi}{\sin n\pi}\int x^\alpha\,\phi(\alpha)\,d\alpha,$$

or
$$\int_0^x \frac{da}{(x-a)^{1-n}} \int_0^a \frac{f'(z)\,dz}{(a-z)^n} = \frac{\pi}{\sin n\pi} f(x).$$

Now if the datum of the problem had been

$$\phi(a) = \int_0^a \frac{ds}{(a-x)^n},$$ we should have had

$$\int_0^x \frac{\phi(a)\,da}{(x-a)^{1-n}} = \int_0^x \frac{da}{(x-a)^{1-n}} \int_0^a \frac{ds}{(a-x)^n}$$

$$= \frac{\pi}{\sin n\pi} s.$$

Hence, as in the given problem we have $n = \frac{1}{2}$,

$$s = \frac{1}{\pi}\int_0^x \frac{\phi(a)\,da}{(x-a)^{\frac{1}{2}}},$$

which is the required expression. (ABEL, *Œuvres*, Tom. I.)

Ex. I. Suppose the Tautochrone be required

$$\phi'(a) = \sqrt{(2g)}\, t_0.$$

Here $$s = \frac{\sqrt{(2g)}\, t_0}{\pi} \int_0^x \frac{da}{(x-a)^{\frac{1}{2}}} = \frac{2\sqrt{(2g)}\, t_0}{\pi} x^{\frac{1}{2}};$$

or $s^2 = \dfrac{8gt_0^2}{\pi^2} x$; the cycloid, as in § 173.

Ex. II. Let $\phi(a) = \sqrt{(2g)}\, \dfrac{a}{c}$, that is let the time be proportional to the vertical height fallen through.

Here $\dfrac{\pi cs}{\sqrt{(2g)}} = \displaystyle\int_0^x \frac{ada}{(x-a)^{\frac{1}{2}}} = x^{\frac{3}{2}} \frac{\Gamma(2)\,\Gamma(\frac{1}{2})}{\Gamma(2+\frac{1}{2})} = \frac{4}{3} x^{\frac{3}{2}}$, the equation to the required curve.

THE END.

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